EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 30 April 2025   14.00 - 15.40

Module 3F1

SIGNALS & SYSTEMS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A floating wind turbine is susceptible to negative damping due to the interaction between rotor thrust, nacelle pitch and surge motion. The motion of the turbine is described by the following set of equations

$$
\begin{aligned}
\theta(t) &= \theta_0 + \theta_{\mathrm{var}}(t) \\
\theta_{\mathrm{var}}(t) &= f_0 \dot{y} \cos(\phi) + \frac{f_0}{\omega} \ddot{y} \sin(\phi) \\
\theta(t) &= m\ddot{y} + c\dot{y} + ky
\end{aligned}
$$

where $\theta(t)$ is the thrust, $y(t)$ is the position, ω is the motion eigenfrequency, ϕ is the phase difference between nacelle velocity and thrust, m is the mass of the nacelle, and θ_0, f_0, c and k are positive system constants. A hydraulic actuator in the tower anchoring is to be used to control the system by countering the thrust, in such a manner that θ_0 is no longer constant but satisfies $\theta_0(t) = \ddot{x} - a\dot{x}$, where $x(t)$ is the actuator input signal and a is a constant.

(a) Show that the transfer function in the Laplace domain from actuator input $\bar{x}(s)$ to position $\bar{y}(s)$ is

$$
G(s) = \frac{s^2 - as}{\left(m - \frac{f_0}{\omega}\sin(\phi)\right)s^2 + (c - f_0\cos(\phi))\,s + k}
$$

and deduce the condition on ϕ for negative damping. [10%]

(b) In order to design a digital controller for the wind turbine, a discrete-time model of the system is needed. The position $y(t)$ is measured and sampled with period $T = 1$ second, and the actuator input signal $x(t)$ is generated by a Digital to Analog (D/A) converter from a discrete-time signal $\{x_k\}$ such that $x(t)$ remains constant over each sample period T. Using a response-invariant method appropriate for this D/A converter, derive the discrete-time model

$$
G(z) = \frac{z^2 - \left(1 - \frac{\sqrt{3}}{2}\right)z - \frac{\sqrt{3}}{2}}{z^2 + \sqrt{3}z + 3}
$$

for $\phi = 0$, $m = \omega = 1$, $f_0 - c = 2a = \log_e(3)$, and $k = \left(\frac{\log_e(3)}{2}\right)^2 + \left(\frac{2\pi}{3}\right)^2$. [40%]

(c) Draw a pole-zero diagram of $G(z)$ and comment on the stability of the system. [10%]

(d) We wish to investigate whether the system $G(z)$ can be stabilized in a feedback loop using a proportional controller K. A partial Nyquist diagram of the open loop system is given in Fig. 1, where the normalised frequency θ goes from 0 to π. The diagram crosses the real axis for $\theta = 0$, $\theta = 1.96422$ and $\theta = \pi$.

(i) Sketch the complete Nyquist diagram.

(ii) Determine for which values of K, if any, the feedback loop achieves stability.

[40%]

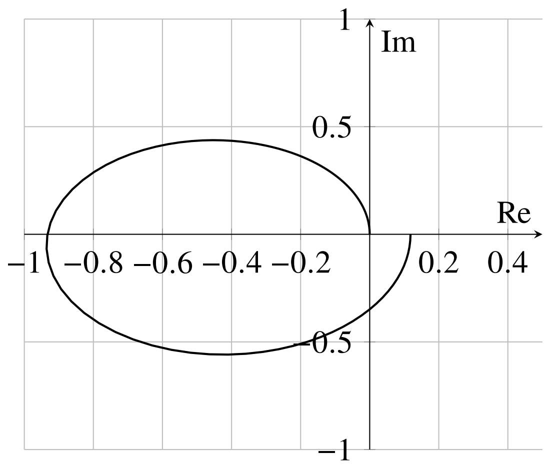

Fig. 1

2 The delta response of a system is

$$g_k = \begin{cases} 1 & \text{for } k = 0 \\ -\frac{1}{2} & \text{for } k = 1 \\ \frac{1}{2}\left(\frac{1}{2}\right)^{k-1} & \text{for } k \geq 2 \end{cases}$$

(a) State the transfer function $G(z)$ of the system in the z domain. [20%]

(b) Draw the system's pole-zero diagram. [20%]

(c) The system maps input signals $\{x_k\}_{k \geq 0}$ to output signals $\{y_k\}_{k \geq 0}$. State the difference equation that relates input signal to output signal. [20%]

(d) Using only the delta response and not the transfer function, verify that the system is stable. [15%]

(e) Let $\{w_k\}_{k \geq 0}$ be the system's response to a unit step $u_k = \begin{cases} 1 \text{ for } k \geq 0 \\ 0 \text{ for } k < 0 \end{cases}$

Compute $\lim_{k \to \infty} w_k$. [10%]

(f) State the stationary (steady-state) response to the input $\{a_k\}_{k \geq 0} = \{\sin(\pi k/2)\}$. [15%]

3 (a) Consider the vector $\mathbf{x} = [1, 2, 3, 4, 1, 4, 3, 2]$. A filter $\mathbf{F}$ is applied to the vector in the frequency domain as follows: the 8 point Discrete Fourier Transform (DFT) $\mathbf{X}$ of $\mathbf{x}$ is computed, then multiplied elementwise by the filter $\mathbf{F}$, i.e., $Y_k = X_k \cdot F_k$ for $k = 0, 1, \ldots, 7$, then the inverse DFT $\mathbf{y}$ of the resulting vector $\mathbf{Y}$ is computed. Below, we have a list of filters $\mathbf{F}$ on the left and a list of filtered vectors $\mathbf{y}$ on the right, with components rounded to one significant digit.

Filter	Filtered output
$\mathbf{F_1} = [1, 0, 2, 0, 3, 0, 4, 0]$	$\mathbf{y_a} = [-0.1, \frac{3}{2}, 5.9, \frac{11}{2}, -2.9, \frac{15}{2}, 3.1, -\frac{1}{2}]$
$\mathbf{F_2} = [1, 2+j, 3+j, 4+j, 5, 6+j, 7+j, 8+j]$	$\mathbf{y_b} = [0, -\frac{5}{2}-\frac{1}{2}j, -1.4j, \frac{5}{2}-\frac{1}{2}j, 0, \frac{5}{2}+\frac{1}{2}j, 1.4j, -\frac{5}{2}+\frac{1}{2}j]$
$\mathbf{F_3} = [1, -1, -2, 3, 2, 3, -2, -1]$	$\mathbf{y_c} = [-2, 4+j, 4, 4-j, -2, 4+j, 4, 4-j]$
$\mathbf{F_4} = [1, 2+j, 3+j, 4+j, 2, 4-j, 3-j, 2-j]$	$\mathbf{y_d} = [-1, j, 5, 6+j, -1, 6-j, 5, -j]$
$\mathbf{F_5} = [1, 2, 3, 4, 1, 2, 3, 4]$	$\mathbf{y_e} = [-5-j, 2j, 5+3.8j, 10, -5-j, 10+2j, 5-1.8j, -4j]$
$\mathbf{F_6} = [0, 4, 0, 3, 0, 2, 0, 1]$	$\mathbf{y_f} = [6.3, \frac{5}{2}, -\frac{1}{2}, \frac{9}{2}, 0.7, \frac{9}{2}, -\frac{1}{2}, \frac{5}{2}]$

Determine which filtered vector corresponds to which filter and justify your choice based on the properties of the DFT. [50%]

(b) Consider a real-valued, possibly double-sided signal $\{x_k\}$, i.e., the signal may have non-zero values at negative times. Let $X(\theta)$ be the Discrete-Time Fourier Transform (DTFT) of $\{x_k\}$.

(i) If $X(\pi/3) = 1.2+j$, what is the value of $X(5\pi/3)$? [10%]

(ii) What condition would $\{x_k\}$ need to fulfil for $X(\theta)$ to be real for all θ? [10%]

For each of the following time-domain operations on $\{x_k\}$ yielding an output signal $\{y_k\}$, explain how $X(\theta)$ is mapped to the DTFT $Y(\theta)$ of the output signal.

(iii) $\{y_k\}$ is the signal time-shifted by d, i.e., $y_k = x_{k-d}$ for all k. [10%]

(iv) $\{y_k\}$ is obtained by discrete-time convolution $y_k = \sum_{\ell=-\infty}^{\infty} g_\ell x_{k-\ell}$ for all k, where $\{g_k\}$ is a (possibly two-sided) signal with DTFT $G(\theta)$. [10%]

(v) $\{y_k\}$ is the elementwise multiplication with $\{g_k\}$, i.e., $y_k = g_k x_k$ for all k. [10%]

4 Consider a simplified model of a twisted pair transmission line drawn in Fig. 2. Output voltage $Y(t)$ and input voltage $X(t)$ satisfy the differential equation

$$Y(t) + \frac{L}{R}\frac{dY(t)}{dt} = X(t)$$

(a) Let $X(t)$ be a zero-mean white noise process with $S_X(\omega) = N_0$ for all ω. Determine the auto-correlation function of the output voltage $Y(t)$ in terms of L/R and N_0, and find the mean power $E[Y^2(t)]$ of the output signal. [40%]

Hint: the Fourier transform of $e^{-a|t|}$ is $\frac{2a}{a^2+\omega^2}$ for $a > 0$.

(b) Explain why a zero-mean white noise input process can only be an approximation of a true physical system. [10%]

(c) A more realistic scenario models $X(t)$ as bandlimited noise, which is zero-mean white noise subjected to an ideal low-pass filter with cutoff frequencies $\pm f_0$ Hz. Express the ratio of output to input power as a function of L/R, N_0 and f_0. [50%]

Hint: use the substitution $\omega = \frac{R}{L}\tan\theta$ suggested in the Mathematics Data Book.

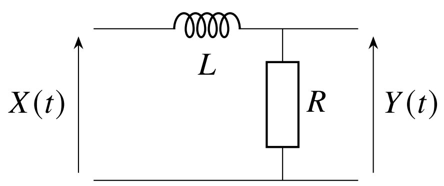

Fig. 2

END OF PAPER

No answers are provided for the 3F1 exam as the assessor does not believe in the value of working to an answer. If you are unsure about your answer, please consult the answers in the crib provided and the methods used to obtain them (which may not be unique…)