3F1 – Crib

1. (a)

$$G(z) = \frac{K\left(z - e^{j\frac{\pi}{4}}\right)\left(z - e^{-j\frac{\pi}{4}}\right)}{z^2}$$

$$= K\,\frac{z^2 - \left(e^{j\frac{\pi}{4}} + e^{-j\frac{\pi}{4}}\right)z + 1}{z^2}$$

$$= K\left(1 - \sqrt{2}\,z^{-1} + z^{-2}\right)$$

and $G(1) = 1 \iff K = \dfrac{1}{2-\sqrt{2}}$

Difference equation:

$$y_k = K\left(u_k - \sqrt{2}\,u_{k-1} + u_{k-2}\right)$$

(b)

$$y_0 = K\,0 = 0$$

$$y_1 = K \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2(2-\sqrt{2})}$$

$$y_2 = K\left(\sin\frac{\pi}{2} - \sqrt{2}\,\sin\frac{\pi}{4}\right) = 0$$

$$y_k = 0, \quad k \geq 2$$

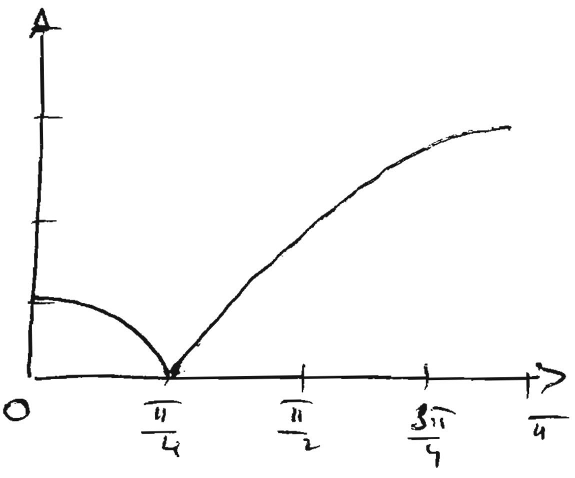
Steady-state response is zero because $G\left(e^{j\frac{\pi}{4}}\right) = 0$ – It is reached after two steps because of the two poles at zero (dead-beat response)

(c) $$G(e^{j\theta}) = K(1 - \sqrt{2}\, e^{-j\theta} + e^{-2j\theta})$$

$$= K e^{-j\theta}(e^{j\theta} + e^{-j\theta} - \sqrt{2})$$

$$= K e^{-j\theta}(2\cos\theta - \sqrt{2})$$

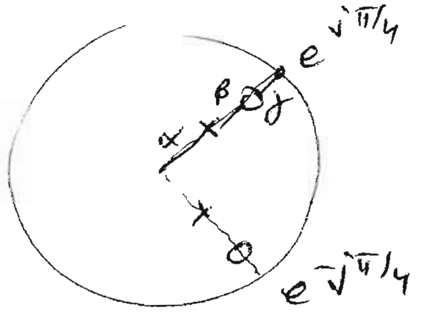
$$\Rightarrow \quad |G(e^{j\theta})| = \left| \frac{2\cos\theta - \sqrt{2}}{2 - \sqrt{2}} \right|$$



Robustness: finite impulse response $\Rightarrow$ No stability issue

Performance: large deviation of the gain from unity gain. Lack of localization of notch.

Alternative design (cf. slides Handout 2)



$\alpha + \beta + \gamma = 1$

minimize α for robustness

minimize β for localization of notch

minimize γ for maximal gain attenuation at 50Hz (a1) $\quad s \to \omega_0 \dfrac{z-1}{z+1} \iff z = \dfrac{\omega_0+s}{\omega_0-s}$

$$\cancel{G} \; H(s) = G\left(\frac{\omega_0+s}{\omega_0-s}\right)$$

$$= K\left(\frac{(\omega_0+s)^2 - \sqrt{2}\,(\omega_0^2-s^2) + (\omega_0-s)^2}{(\omega_0+s)^2}\right)$$

$$= K \frac{(2-\sqrt{2})\,\omega_0^2 + (2+\sqrt{2})\,s^2}{(\omega_0+s)^2}$$

$$= \frac{\omega_0^2 + \frac{2+\sqrt{2}}{2-\sqrt{2}}\,s^2}{(\omega_0+s)^2}$$

2 zeros at $s = \pm j\,\omega_0\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$

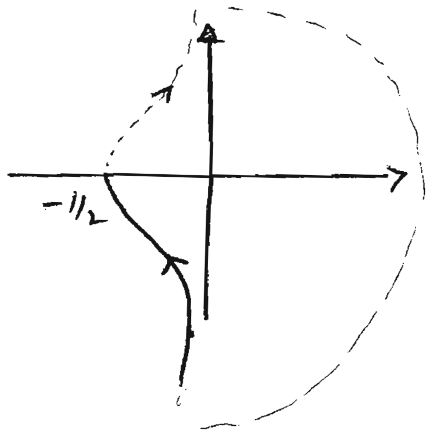
2 poles at $s = -\omega_0$

unity DC gain

2. (a)

(i) See slide 31

(ii)



(iii) Stability $\Leftrightarrow$ no encirclement of $-\frac{1}{k}$

$\Leftrightarrow \quad -\frac{1}{k} < -\frac{1}{2}$

$\Leftrightarrow \quad 0 < k < 2$

(iv) Closed-loop transfer function:

$$\frac{G(z)}{1 + k\,G(z)}$$

poles of $\frac{G(z)}{1+k\,G(z)}$ are roots of

$$z - 1 + k = 0$$

$$\Rightarrow \quad z = 1 - k$$

z is inside the unit circle for

$$0 < k < 2 .$$

2.b.

(i) $F_Z(z)$ is $\Pr\{Z \le z\} = \Pr\{\max(X, Y) \le z\}$

$= \Pr\{X \le z \text{ and } Y \le z\}$

$= F_{XY}(z, z)$

(ii) If $X \perp Y$, then $F_Z(z) = F_{XY}(z, z) = F_X(z) F_Y($

$$f_Z(\cancel{z}) = \frac{d}{dz} F_Z(\cancel{z}) = \frac{d}{dz} F_X(z) F_Y(z)$$

$$= f_X(z) F_Y(z) + f_Y(z) F_X(z)$$

(iii) $f_{X,Y}(x, y) = f_X(x) f_Y(y)$, hence $X \perp Y$

$$F_X(x) = \int_0^x \alpha e^{-\alpha w} dw = (1 - e^{-\alpha x})$$

$$F_Y(y) = 1 - e^{-\beta y}$$

So

$$F_Z(z) = \begin{cases} (1 - e^{\alpha z})(1 - e^{-\beta z}) & z \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Z(z) = \begin{cases} \alpha e^{-\alpha z}(1 - e^{-\beta z}) + \beta e^{-\beta z}(1 - e^{-} & z \\ 0 & \text{otherwise} \end{cases}$$

3.

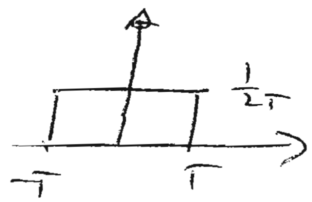
(a) Note: Stationary $\Rightarrow$ WSS

(b) (i) $E X(t) = 0$

(ii) $r_{xx}(t_1, t_2) = \frac{1}{2} E(z^2) \cos \omega (t_1 - t_2)$

(iii) WSS ($E X(t) = 0$ & $r_{xx}(t, t+\tau) = r_{xx}(\tau)$)

(c) (i) $h(t) = \begin{cases} \frac{1}{2T} & -T \le t \le T \\ 0 & \text{otherwise} \end{cases}$



$$H(\omega) = \frac{\sin \omega T}{\omega T}$$

(ii) For X WSS, $S_Y(\omega) = H^*(\omega) H(\omega) S_X(\omega)$

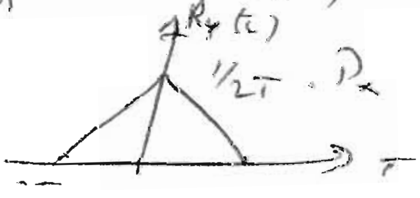
$$= \frac{\sin^2 \omega T}{\omega^2 T^2} S_X(\omega)$$

(iii) For $r_{xx}(\tau) = P_x \delta(\tau)$, $S_x = P_x$

$$\therefore \quad S_Y(\omega) = P_x \frac{\sin^2 \omega T}{\omega^2 T^2}$$

$$R_Y(\tau) = \frac{1}{2\pi} \int S_Y(\omega) e^{j\omega\tau} d\omega$$

Since $S_Y(\omega) = H^*(\omega) H(\omega) P_x$, R_Y is the convolution of $h(t)$ with itself



4.

(a) The codeword lengths are computed as $L_x = \lceil -\log_2 P_X(x) \rceil$, i.e.,

x	L_x
A	4
B	3
C	5
D	2
E	4
F	5
G	3

(b) Shannon's contruction is given below, using the following steps

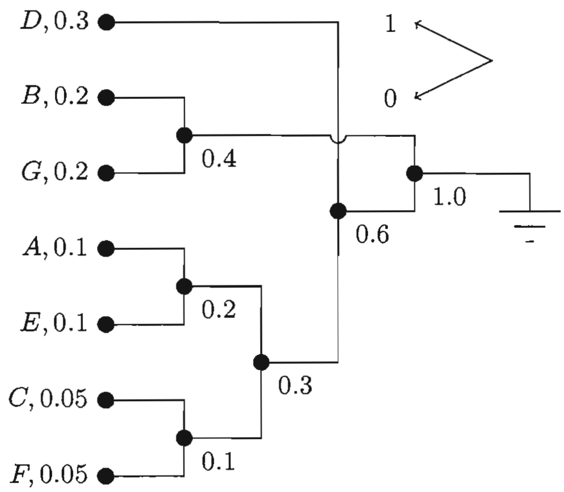
(a) order the alphabet in order of decreasing probabilities
(b) compute the cumulative probability values for each symbol
(c) express the cumulative probabilities in binary
(d) truncate the cumulative probability values at the specified codeword lengths

x	$P_X(x)$	$F_X(x)$	L_x	$F_X(x)$ in binary	Codeword
D	0.3	0	2	0.**00**0000...	00
B	0.2	0.3	3	0.**010**011...	010
G	0.2	0.5	3	0.**100**000...	100
A	0.1	0.7	4	0.**1011**00...	1011
E	0.1	0.8	4	0.**1100**11...	1100
C	0.05	0.9	5	0.**11100**1...	11100
F	0.05	0.95	5	0.**11110**0...	11110

The expected codeword length is

$$E[L_{SF}] = \sum_x L_x P_X(x) = 3.1$$

(c) The Huffman algorithm yields the tree below. Note that we have ordered the symbols in decreasing probabilities for ease of representation.



The codewords are read back from the tree to yield

x	Codeword
D	10
B	00
G	01
A	1100
E	1101
C	1110
F	1111

The expected codeword length can either be evaluated using the expression $E[L_H] = \sum_x L_x P_X(x)$ or using the "path length lemma" summing the labels on all the intermediate nodes in the tree above, i.e.,

$$E[L_H] = 1.0 + 0.6 + 0.4 + 0.3 + 0.2 + 0.1 = 2.6$$

(d) The entropy is calculated using the expression

$$H(X) = -\sum_x P_X(x) \log_2 P_X(x) = 2.54644 \text{ [bits]}$$

We observe that $E[L_H]$ is within about 0.05 bits of entropy, whereas $E[L_{SF}]$ is within about 0.55 bits of entropy.

The following relation always holds

$$H(X) \leq E[L_H] \leq E[L_{SF}] < H(X) + 1$$

with the last inequality strict. The second inequality holds because the Huffman code is the optimal prefix-free code for a random variable.

(e) No, it is not possible, because these lengths do not satisfy Kraft's inequality, i.e.,

$$\sum_y 2^{-L_y} = 3 \times 2^{-2} + 2 \times 2^{-3} + 2^{-4} = 1.0625 > 1$$