1 a) (i) Let α, β be scalars and let x_k, y_k be discrete time signals. Then

$$\mathcal{Z}[\alpha x_k + \beta y_k] = \alpha \mathcal{Z}[x_k] + \beta \mathcal{Z}[y_k].$$

(ii) From lectures : $\mathcal{Z}[k a^k] = \dfrac{az^{-1}}{(1-az^{-1})^2}$

Using linearity of $\frac{d}{da}$:

$$\frac{d}{da} \mathcal{Z}[k a^k] = \frac{d}{da} \frac{az^{-1}}{(1-az^{-1})^2}$$

(b ii cont.)

$$\mathcal{Z}[k^2 a^{k-1}] = \frac{(1-az^{-1})^2 z^{-1} + 2az^{-2}(1-az^{-1})}{(1-az^{-1})^4}$$

$$= \frac{z^{-1} - az^{-2} + 2az^{-2}}{(1-az^{-1})^3}$$

$$= \frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$$

Therefore, by linearity

$$\mathcal{Z}[k^2 a^k] = \frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$$

b) i) We have $\ddot{Y} + \dot{Y} + Y = F(t)$.

Taking Fourier Transform:

$$[(j\omega)^2 + j\omega + 1]\bar{Y}(\omega) = \bar{F}(\omega)$$

$$\bar{Y}(\omega) = \frac{1}{1-\omega^2+j\omega}\bar{F}(\omega).$$

This gives P.S.D.s:

$$S_Y(\omega) = \left|\frac{1}{1-\omega^2+j\omega}\right|^2 S_F(\omega)$$

$$= \frac{1}{(1-\omega^2)^2+\omega^2} S_F(\omega)$$

$$= \frac{1}{1-\omega^2+\omega^4} S_F(\omega) \longrightarrow 0 \quad \text{as } \omega \to \infty.$$

Therefore lower bound of average power of Y is 0.
There is no input that can achieve this because it would require all of the power to be concentrated above an arbitrarily high frequency.

(ii) Using above calculation, $\max_\omega S_Y(\omega)$ occurs at minimum of $1-\omega^2+\omega^4$.

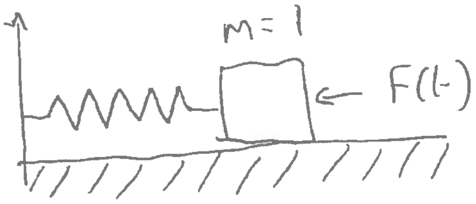
Let $x = \omega^2$. $\frac{d}{dx}(1-x+x^2) = -1+2x = 0 \Rightarrow x = 1/2$.

Therefore $\max_\omega S_Y(\omega) = \frac{1}{1-\frac{1}{2}+\frac{1}{4}} S_F(\omega) = \frac{4}{3} S_F(\omega)$

This is achieved when $\omega = 1/\sqrt{2}$.

An example is $F(t) = \sqrt{2}\cos\left(\frac{2\pi}{\sqrt{2}}t + \theta\right)$ where θ is uniformly distriubled on $[0, 2\pi)$.

(iii) G can be realised by an (underdamped) simple harmonic oscillator, e.g. a mass-spring system with friction:



~~The~~ physical relevance of (ii) is

- the total power of any physical system is bounded
- this particular system exhibits resonance, a peak in power output at a particular input frequency.

Question 2

(a) Given the sampling frequency $f = 44.1$ KHz and the cut-off frequency of $\omega_\ell = 500$Hz and $\omega_h = 4$KHz the normalized cut-off frequency reads

$$\theta_\ell = \omega_\ell T \pi = (\omega_\ell/f)\pi \simeq 0.0356 \qquad \theta_h = (\omega_h/f)\pi \simeq 0.3562$$

For the low pass filter A, let L be the ideal low-pass filter with normalized cut-off frequency θ_c. By Fourier anti-transform, $\mathcal{F}^{-1}$,

$$\ell_k = \mathcal{F}^{-1}(L) = \frac{1}{2\pi}\int_{-\pi}^{\pi} L(e^{j\theta})e^{j\theta k}d\theta = \frac{1}{2\pi}\int_{-\theta_\ell}^{\theta_\ell} e^{j\theta k}d\theta = \frac{1}{2\pi}\frac{e^{j\theta_\ell k} - e^{-j\theta_\ell k}}{jk} = \frac{\sin(\theta_\ell k)}{\pi k}.$$

Rectangular windows corresponds to a simple truncation. For $M = 100$, causality is then achieved via a shifting of 50 samples. Combined these operations lead to the FIR filter with impulse response

$$a_k = \frac{\sin(\theta_\ell(k-50))}{\pi(k-50)} \qquad 0 \le k \le M = 100\ .$$

For B, we take the ideal filter $H = 1 - L$. By Fourier anti-transform and linearity,

$$h_k = \mathcal{F}^{-1}(1-H) = \mathcal{F}^{-1}(1) - \mathcal{F}^{-1}(H) = \delta_k - \frac{\sin(\omega_c k)}{\pi k}$$

where δ_k is the unit pulse signal. So, as above,

$$b_k = \delta_{k-50} - \frac{\sin(\theta_h(k-50))}{\pi(k-50)} \qquad 0 \le k \le M = 100\ .$$

(b) A generic FIR filter G has linear phase $G(e^{j\theta}) = |G(e^{j\theta})|e^{-j\theta\frac{M}{2}}$ whenever it is symmetric, i.e. $g_k = g_{M-k}$. So, A and B have linear phase, starting at 0 and ending at $-\frac{\pi M}{2}$, as sketched below.

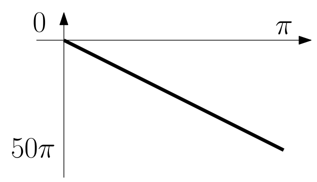


Figure 1: The linear phase of the filters A and B.

(c) Given the normalized cut-off frequency above, the equivalent pre-warped analogue filter cutoff frequency corresponds to

$$\bar{\theta}_\ell = \tan(\theta_\ell/2) = \tan(0.0178) \simeq 0.0178 \qquad \bar{\theta}_h = \tan(\theta_h/2) = \tan(0.1781) \simeq 0.1800\ .$$

Using the suggested band transformation and the bilinear transform, the band-pass filter is given by

$$C(z) = \frac{1}{\frac{\left(\frac{z-1}{z+1}\right)^2 + \bar{\theta}_\ell\bar{\theta}_h}{\left(\frac{z-1}{z+1}\right)(\bar{\theta}_h - \bar{\theta}_\ell)} + 1}\ .$$

Students should simplify this expression as much as possible.

(d) We can hear sounds up to $\omega_{max} = 20$KHz. From Shannon theorem, the sampling frequency must be greater than $2\omega_{max} = 40$KHz. This guarantees that the original signals can be reconstructed after sampling without distortion (at least in theory). The sampling frequency f satisfies $f \geq \omega_0$ so it is adequate. The reason why f is taken slightly larger than 40KHz is justified by the use of non-ideal filters at reconstruction, to take into account of the filters' transition band.

$f_\ell = f/2 < 2\omega_{max}$ therefore it does not satisfy Shannon theorem. With that sampling frequency there are sounds that we would not be able to reconstruct. Reconstructed signals would be affected by aliasing, that is, by frequency distortion. In the time domain this means that sampling would map different sounds into the same sequence of samples, making them indistinguishable at reconstruction stage.

$f_h = 2f > 2\omega_{max}$ satisfies Shannon theorem. This frequency is adequate for sampling. However, in comparison to f, sampling at frequency f_h would double the amount of data, requiring more storage. This makes f_h non optimal.

(e) A and B are FIR filters with horizon $M = 100$ and a N-points FFT hardware with $N = 1024$. Let's use g_k to denote the impulse response of each filter, and x_k to denote the input samples to the filters.

1. set the filter into an array of 1024 elements $G = [g_0\ g_1\ \dots\ g_M\ 0\ \dots\ 0]$.
2. Organise the input samples into arrays of 1024 elements

   - $X_0 = [\ \underbrace{0\ \dots\ 0}_{M \text{ samples}}\ |\underbrace{x_0\ \dots\ x_{N-M-1}}_{N-M \text{ samples}}]$
   - $X_1 = [\ \underbrace{x_{N-M-1-M}\ \dots\ x_{N-M-1}}_{M \text{ last samples from the previous frame}}\ |\underbrace{x_{N-M}\ \dots\ x_{2(N-M)-1}}_{N-M \text{ new samples}}]$
   - $\dots$
   - $X_k = [\ \underbrace{*\ \dots\ *}_{M \text{ last samples from previous frame}}\ |\ \underbrace{*\ \dots\ *}_{N-M \text{ new samples}}\ ]$.

3. Apply FFT: $\bar{G} = \mathbf{FFT}(G)$, $\bar{X}_k = \mathbf{FFT}(X_k)$ for all $k \geq 0$.
4. Compute the output arrays $\bar{Y}_k = GX_k$, for all $k \geq 0$.
5. Apply inverse FFT: $Y_k = \frac{1}{N}\mathbf{FFT}(\bar{Y}_k^*)*$ for all $k \geq 0$.
6. Reconstruct the filter output by extracting data from each array. For all $k \geq 0$,

$$Y_k = [\ \underbrace{*\ \dots\ *}_{\text{remove the first } M \text{ samples}}\ |\ \underbrace{*\ \dots\ *}_{\text{collect } N-M \text{ samples respecting their order}}\ ].$$

The collected samples correspond to the output samples y_k of the filter.

Question 3

(a) Step response invariance corresponds to the following operations

$$P(z) = \frac{z-1}{z}\mathcal{Z}\left(\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)_{t=kT}\right) .$$

Specifically, using partial fractions,

$$\frac{G(s)}{s} = \frac{A}{s} + \frac{B}{s+d}$$

where $A = 1/d$ and $B = -1/d$. Then, Laplace anti-transform and sampling lead to

$$\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)_{t=kT} = \frac{1}{d}\left(1 - e^{-dTk}\right) .$$

Applying the $\mathcal{Z}$ transform gives

$$\mathcal{Z}\left(\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)_{t=kT}\right) = \frac{1}{d}\left(\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-dT}z^{-1}}\right)$$

Finally, multiplication by $\frac{z-1}{z} = 1 - z^{-1}$ gives

$$\frac{1}{d}\left(1 - \frac{1-z^{-1}}{1-e^{-dT}z^{-1}}\right) = \frac{1}{d}\left(1 - \frac{z-1}{z-e^{-dT}}\right) = P(z) .$$

Stability is preserved for any $T > 0$ since $P(z)$ has a pole in e^{-dT} which is inside the unit circle for any $T > 0$.

(b) The right plot is (b). $P(z)$ has one stable pole, therefore the magnitude plot is monotonically decreasing. This is not satisfied by diagram (a).

The right phase plot is (c). $P(z)$ has one stable pole, therefore the phase goes from 0 to -180 degrees. This is not satisfied by diagram (d).

For $\omega \to 0$ the frequency the magnitude plot is at 0 dB therefore $d = 1$.

d modulates the gain of the transfer function $P(z)$: for larger (smaller) d, the magnitude plot decreases (increases) of a factor $\frac{1}{d}$, that is, $-20\log(d)$. The pole in $e^{-dT} = e^{-0.02d}$ moves closer to zero (closer to the unit circle) as d increases (decreases). As a consequence, the system shows faster (slower) transients, thus the roll-off moves at higher (lower) frequencies.

(c) The discretized input reads

$$u(k) = 3 + w\sin(\omega k) .$$

For $T = 0.02$, using linearity, the steady-state response of the car is

$$y = 3|P(0)| + 2|P(e^{j\omega T})|\sin(\omega k + \angle P(e^{j\omega T})) .$$

From the Bode diagrams in Figure 1(b) and in Figure(c), $|P(0)| = 0\text{dB} = 1$, $|P(e^{j\omega T})| = |P(e^{j0.02})| = -3\text{dB} = 10^{-3/20} \simeq 0.7079$, and $\angle P(e^{j\omega T}) = -\frac{\pi}{4}$.

For larger ω, $|P(e^{j\omega T})|$ reduces therefore the oscillations of the output speed become less noticeable.

(d) The correct Nyquist diagram is the left one: $K(z)$ has a pole on the unit circle, therefore the Nyquist diagram must have an asymptote. The completed Nyquist diagram is in Figure 2.

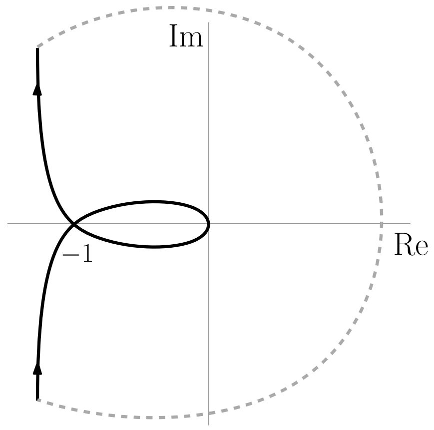


Figure 2: The complete Nyquist diagram

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There are no unstable poles in open loop. Thus, from the Nyquist criterion, closed-loop stability requires a Nyquist locus with 0 encirclements of the point $\frac{-1}{k}$. It follows that the closed loop is stable for any $0 \leq k < 1$.

4 (a)(i) $y_k = 1, \underbrace{0, \dots 0}_{n}, 1, \dots$

$$\bar{y}(z) = 1 + z^{-n} + z^{-2n} + \dots$$

$$= \frac{1}{1-z^{-n}}$$

Let u_k be unit step. Then $\bar{u}(z) = \frac{1}{1-z^{-1}}$

Therefore $\bar{u}(z)\bar{g}(z) = \bar{y}(z)$

$$\Rightarrow \bar{g}(z) = \frac{1-z^{-1}}{1-z^{-n}} = \frac{1}{1-z^{-n}} - \frac{z^{-1}}{1-z^{-n}}$$

So pulse response, g_k, is

$$g_k = \underbrace{1, -1, 0 \dots 0}_{n}, 1, -1, 0, \dots \quad \text{etc.}$$

$$= \begin{cases} 1 & \text{if } k \bmod n = 0 \\ -1 & \text{if } k \bmod n = 1 \\ 0 & \text{otherwise} \end{cases}$$

(ii) G is not BIBO stable because it has poles on the unit circle, $|z_p| = 1$.

(iii) From (i): $\frac{\bar{y}(z)}{\bar{u}(z)} = \bar{g}(z) = \frac{1-z^{-1}}{1-z^{-n}}$

$$(1-z^{-n})\,\bar{y}(z) = (1-z^{-1})\,\bar{u}(z)$$

Required difference equation is:

$$y_k - y_{k-n} = u_k - u_{k-1}$$

with $y_0 = y_{-1} = \dots = y_{-n} = 0$.

b (i) $\{X(t)\}$ is WSS and $\mathcal{H}$ is L.T.I. therefore $\{Y(t)\}$ is WSS

(ii) From block diagram we have:

$$y(t) = x(t) - x(t-\Delta)$$

Using Fourier transform table, freq. response $\mathcal{H}(\omega)$ is given by:

$$\mathcal{H}(\omega) = 1 - e^{-j\omega\Delta}$$

$$\therefore |\mathcal{H}(\omega)|^2 = (1 - e^{-j\omega\Delta})(1 - e^{+j\omega\Delta})$$

$$= 1 - e^{j\omega\Delta} - e^{-j\omega\Delta} + 1$$

$$\left.\begin{aligned} &= 2 - 2\cos(\omega\Delta) \\ &= 2\sin^2\left(\frac{\omega\Delta}{2}\right) \end{aligned}\right\} \text{either form fine.}$$

Now $S_y(\omega) = |\mathcal{H}(\omega)|^2 S_x(\omega)$

$$\therefore S_y(\omega) = 2\sin^2\left(\frac{\omega\Delta}{2}\right) S_x(\omega)$$

(iii) Suppose $r_{XX}(\tau) = P\,\delta(\tau)$. Then $S_x(\omega) = P$.

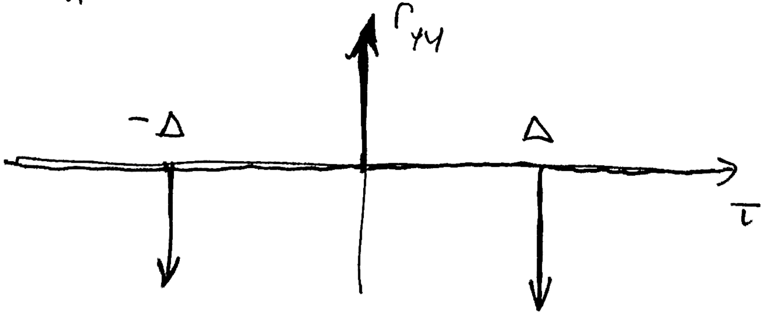
$$S_y(\omega) = 2\sin^2\left(\frac{\omega\Delta}{2}\right) P \qquad \left[= 2P(1 - \cos(\omega\Delta)) \right]$$

$$r_{yy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{2P}{2\pi}\left[\delta(\tau) - \delta(\tau - \Delta) - \delta(\tau + \Delta)\right]$$

$$= \frac{P}{\pi}\left[\delta(\tau) - \delta(\tau-\Delta) - \delta(\tau+\Delta)\right]$$

Sketch:



Examiner's comments

Q1 Z-transform derivation and random process modelling

This question was unpopular and caused the most difficulty. ai) Around one third did not properly define linearity. aii) More than half struggled to derive a Z-transform by differentiating with respect to a, this was analogous to a calculation done in lectures and in the notes. bi-ii) The second part of the question on random processes (second order system driven by white noise) was better answered, with most realising conceptually what was required. biii) many students (around 70%) were unable to describe a physical system that the model might correspond to (e.g. a damped oscillator, RCL circuit, mass-spring-dashpot etc).

Q2 Filter design (FIR, IIR), phase diagrams, Shannon's theorem and aliasing, FFT

A popular question. (a) minor issues with normalized frequency. More mistakes on high-pass than on low-pass filters. (b) is very easy but was not answered by most student; (c) minor issues on frequency warping, (d) well-addressed by most students, (e) most students remembered how to use FFT for filtering but had a few mistakes in organizing data in batches.

Q3 Discretization, Bode plots, steady-state response, Nyquist criterion

A popular question. (a) well answered by most; (b) a few students were confused by the presence of a resonance in Bode diagrams; (c) minor issues in using linearity; (d) well addressed by most of the students with standard mistakes on the use of Nyquist for stability.Q4 Continuous time random processes

Q4 Step response/difference equations, power spectral density

A popular question. ai) The first part was a straightforward calculation to find the step response of a discrete time linear system which was generally well answered. aii) Many students incorrectly stated that poles on the unit circle indicate a stable system. aiii) Many students were unable to derive a difference equation representation of the system. bi) Most answered this correctly, but many performed a long calculation to establish WSS, when all that was needed (for 1 mark) was to state that the system was LTI with WSS input. bii-iii) This involved the calculation of the frequency response of a continuous time LTI system. Most could recall the key relationships derived in lectures and apply them, but many were unable to evaluate a the integral in part (iii) to obtain the autocorrelation function from the power spectral density.