Version RS/5

EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 29 April 2014 9.30 to 11

Module 3F1

SIGNALS AND SYSTEMS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 A digital notch filter is designed to attenuate 50 Hz noise. The sampling period is $T = 2.5$ milliseconds. It is proposed to place two zeros of the transfer function $G(z)$ at $z = e^{\pm j\frac{\pi}{4}}$ and two poles at $z = 0$. It is also desired to have a unity DC gain.

(a) Derive the expression of the transfer function $G(z)$ and of the difference equation that implements the filter. [20%]

(b) Compute the response of the filter for the input signal $u_k = \sin(2\pi 50kT),\ k \geq 0$. Justify your answer. [20%]

(c) Determine the frequency response of the filter and sketch the Bode plots. Explain why it can be said that the proposed filter is robust but not optimal as a notch filter. Explain how the location of poles and zeros could be modified to trade robustness for performance. Illustrate how your proposed modification would affect the frequency response of the filter. [30%]

(d) An alternative design of the digital notch filter would be to apply Tustin transform $s = \frac{2}{T}\frac{z-1}{z+1}$ to the transfer function $H(s)$ of an analog notch filter.

(i) Explain why any *stable* analog filter would be converted into a *stable* digital filter by this method. [20%]

(ii) Determine the transfer function $H(s)$ of the analog filter that would lead to $G(z)$. [10%]

2 (a) (i) State the Nyquist stability criterion for a discrete-time linear system with transfer function $G(z)$ in negative feedback loop with a constant gain controller k. [15%]

(ii) Sketch the Nyquist diagram for the open-loop system

$$G(z) = \frac{1}{z-1}$$

clearly labelling the direction of encirclement. [20%]

(iii) Use the Nyquist stability criterion to determine the values of the feedback gain k for which the closed-loop system is stable. Confirm the result from a direct computation of the closed-loop transfer function. [15%]

(b) Suppose the random variables X and Y are jointly distributed with probability density function (PDF) $f_{X,Y}(x,y)$ and cumulative distribution function (CDF) $F_{X,Y}(x,y) = Pr\{X \leq x \text{ and } Y \leq y\}$.

(i) For $Z = \max(X,Y)$, explain why the CDF of Z is $F_Z(z) = F_{X,Y}(z,z)$. [10%]

(ii) If X and Y are independent, give expressions for the CDF $F_Z(z)$ and the PDF $f_Z(z)$. [20%]

(iii) Suppose X and Y have the PDF

$$f_{X,Y}(x,y) = \begin{cases} \alpha\beta e^{-\alpha x} e^{-\beta y} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $F_Z(z)$ and $f_Z(z)$. [20%]

(TURN OVER

3 (a) Define the terms *strict sense stationary* and *wide sense stationary* (WSS) as applied to random processes, and state which implies the other. [20%]

(b) The deterministic random process $X(t) = r\cos(\omega t + \phi)$ is defined by independent variables r and ϕ. The quantity ϕ is uniformly distributed over $[-\pi, \pi]$ and $E[r^2] = A$.

(i) Find $E[X(t)]$.

(ii) Find the autocorrelation function $r_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$.

(iii) Is $X(t)$ WSS ? Explain.

[30%]

(c) The moving average $Y(t)$ of a process $X(t)$ is calculated as

$$Y(t) = \frac{1}{2T}\int_{t-T}^{t+T} X(\alpha)d\alpha.$$

Consider $Y(t)$ as the output of a linear time invariant system driven by the input $X(t)$.

(i) Give the impulse response $h(t)$ of the LTI system and its transfer function $H(\omega)$. [10%]

(ii) Assume that $X(t)$ is WSS with autocorrelation function $r_{XX}(\tau)$ and power spectral density $S_X(\omega)$. Provide an expression of the PSD $S_Y(\omega)$ of the moving average $Y(t)$ in terms of $H(\omega)$ and $S_X(\omega)$. [20%]

(iii) Suppose $X(t)$ is a white noise process with $r_{XX}(\tau) = P_X\delta(\tau)$. Find $r_{YY}(\tau)$. [20%]

4 A random variable X defined over the alphabet $\{A,B,C,D,E,F,G\}$ has the following probability distribution:

x	A	B	C	D	E	F	G
$P_X(x)$	0.1	0.2	0.05	0.3	0.1	0.05	0.2

(a) Calculate the codeword lengths that a binary Shannon-Fano code would assign to the values of X. [15%]

(b) Design a binary Shannon-Fano code for this random variable and specify the expected codeword length $E[L_{SF}]$ of the code obtained. [20%]

Hint: if you cannot remember Shannon's construction, any prefix-free code with the specified lengths is acceptable as an answer to this question.

(c) Design a Huffman code for this random variable X and hence specify the expected codeword length $E[L_H]$ of the code obtained. [25%]

(d) Compute the entropy $H(X)$ (in bits) of the random variable and compare the expected codeword lengths of your code designs to the entropy. Relate the quantities $E[L_H]$, $H(X)+1$, $H(X)$ and $E[L_{SF}]$ in a sequence of three inequalities that are always true for any probability distribution of X. Which of those inequalities is always strict? [25%]

(e) Is it possible to obtain a prefix-free code with the following codeword lengths for a random variable Y defined over the alphabet $\{A,B,C,D,E,F\}$?

y	A	B	C	D	E	F
L_y	2	2	2	3	3	4

If yes, specify such a code. If no, explain why. [15%]

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