EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 27 April 2021 9.00 to 10.40

Module 3F1

SIGNALS & SYSTEMS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet and at the top of each answer sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed.
You are allowed access to the electronic version of the Engineering Data Books.

10 minutes reading time is allowed for this paper at the start of the exam.

The time taken for scanning/uploading answers is 15 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

1 (a) (i) What does it mean for the z-transform to be linear? [5%]

(ii) Write down the z-transform of $\{ka^k\}_{k\geq 0}$ and use this to compute the z-transform of

$$\{x_k\}_{k\geq 0} = k^2 a^k$$

by exploiting the linearity of the derivative. Express your answer as a simplified rational function. [35%]

(b) Consider a continuous time system, $\mathcal{G}$, with output, Y, and input given by a wide-sense stationary random process, $\{F(t)\}$:

$$\ddot{Y} + \dot{Y} + Y = F(t).$$

Suppose the average power of $\{F(t)\}$ is 1.

(i) What is the lower bound of the average power of Y? Can this bound be achieved? Provide an example of a random process, $\{F(t)\}$, that achieves this bound or a mathematical argument for why such a process does not exist. [20%]

(ii) What is the upper bound of the average power of Y? Can this bound be achieved? Provide an example of a random process, $\{F(t)\}$, that achieves this bound or a mathematical argument for why such a process does not exist. [30%]

(iii) Describe an appropriate physical system that could realise $\mathcal{G}$ and use this to explain the physical relevance of your answer to part (ii). [10%]

2 We design three digital filters for a 3-way speaker, with cutoff frequencies at $\omega_\ell = 500$ Hz and $\omega_h = 5$ kHz. Assume a sampling frequency $f = 44.1$ kHz.

(a) Using a rectangular window, design a FIR low-pass filter A and a FIR high-pass filter B with cutoff frequencies ω_ℓ and ω_h, respectively. Write down the expressions for their impulse responses, a_k and b_k, where $0 \leq k \leq M = 100$. [30%]

(b) Sketch the phase diagrams of the filters A and B. [15%]

(c) Design a IIR band-pass filter C with cutoff frequencies ω_ℓ and ω_h. Use the analogue prototype $\frac{1}{s+1}$, the bilinear transform $s = \frac{z-1}{z+1}$, and the band transformations low-pass to band-pass, $s = \frac{\bar{s}^2+\omega_1\omega_2}{\bar{s}(\omega_2-\omega_1)}$ with lower cutoff ω_1 and upper cutoff ω_2. [20%]

(d) Explain why the use of the sampling frequency $f = 44.1$ kHz is adequate. Could we use a higher sampling frequency, $f_h = 2f$, or a lower sampling frequency, $f_\ell = f/2$? Motivate your answer by considering Shannon's theorem, aliasing, and computational resources. [20%]

(e) Explain how to organise data to compute the output of the FIR filters A and B using a 2^{10}-point FFT hardware. [15%]

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3 We design the cruise-control for a car, which automatically regulates the speed of the vehicle by adapting its throttle f. The (simplified) continuous transfer function $G(s)$ from input throttle f to output velocity v reads

$$G(s) = \frac{1}{s+d}$$

where $d > 0$ is the damping coefficient, modeling energy dissipation.

(a) Using step response invariance and sampling period T, show that the corresponding digital transfer function, $P(z)$, is

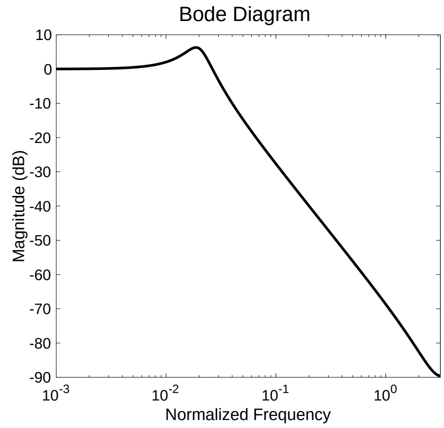
$$P(z) = \frac{1}{d}\left(\frac{1-e^{-dT}}{z-e^{-dT}}\right).$$

[25%]

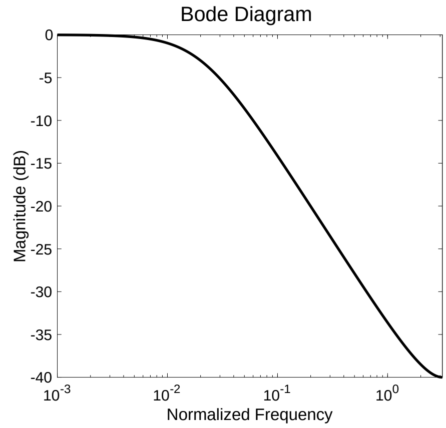
(b) Consider $T = 0.02$. Select the correct magnitude plot and the correct phase plot among the ones in Figure 1 (explain your answer). Compute the value of the parameter d compatible with the magnitude plot. How will the plot change for smaller or larger d? [25%]

(c) Find the steady-state response of the car to slow throttle variations $f = 3 + 2\sin(\omega t)$ where $\omega = 1$ rad/s and t represents time. How will the response change for larger ω? [20%]

(d) Consider the controller $K(z) = \frac{k}{1-z^{-1}}$. Select the correct Nyquist locus of $K(z)P(z)$ between the two in Figure 2 (explain your answer). Sketch the complete Nyquist diagram. Using the Nyquist criterion, determine the range of gains k that guarantees closed loop stability. [30%]



(a)



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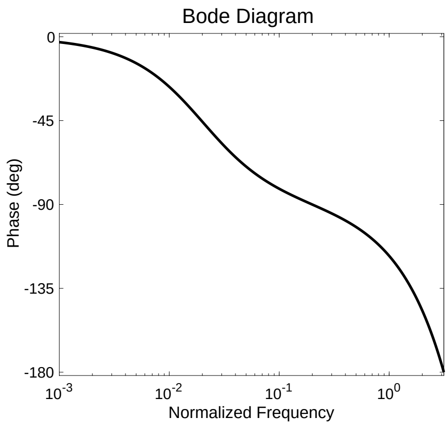

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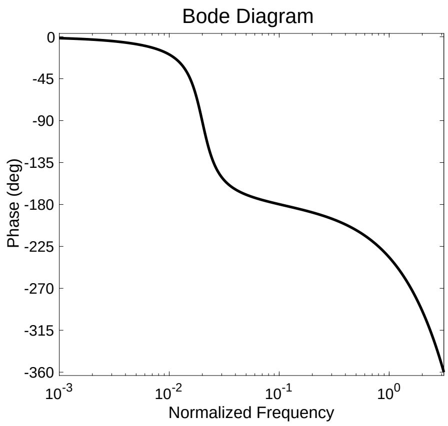
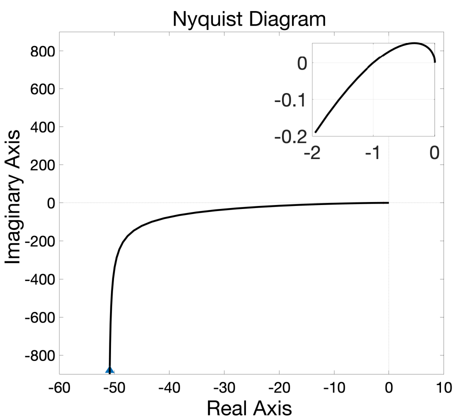
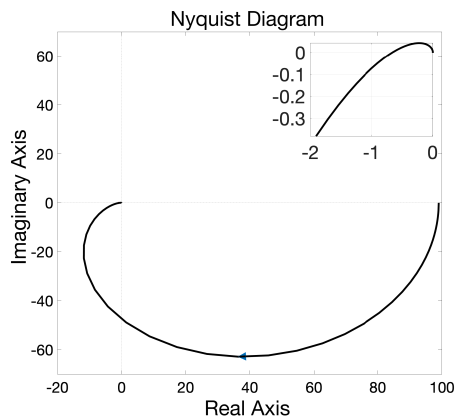


(d)

Fig. 1: Bode diagrams: magnitude and phase plots



(a)



(b)

Fig. 2: Nyquist diagrams

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4 (a) Consider a linear time invariant discrete time system, $\mathcal{G}_n$, with a step response given by:

$$y_k = \underbrace{1, 0, \ldots 0,}_{n} \underbrace{1, 0, \ldots 0,}_{n} 1, 0, \ldots \text{ etc.}$$

(i) Calculate the pulse response, g_k, of $\mathcal{G}_n$. [30%]

(ii) Is $\mathcal{G}_n$ BIBO stable? Justify your answer. [5%]

(iii) Write down a difference equation for $\mathcal{G}_n$ relating an arbitrary input, u_k, to a corresponding output, y_k. [10%]

(b) Consider the continuous time system, $\mathcal{H}$, shown in Figure 3, in which Δ represents a pure delay of Δ time units and the input $\{X(t)\}$ is a wide sense stationary process.

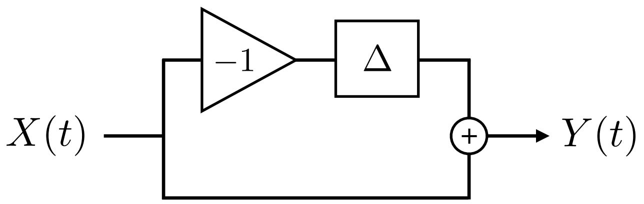

Fig. 3: block diagram of continuous time system $\mathcal{H}$

(i) Will the output, $\{Y(t)\}$, be wide sense stationary? Justify your answer. [5%]

(ii) Suppose the power spectral density of $\{X(t)\}$ is given by $\mathcal{S}_X$. Calculate the power spectral density of the output, $\mathcal{S}_Y$. [30%]

(iii) Now suppose $\{X(t)\}$ is a white noise process with autocorrelation function $r_{XX}(\tau) = P\delta(\tau)$. Calculate and sketch the autocorrelation function, r_{YY}, of $\{Y(t)\}$. [20%]

END OF PAPER

Short/numerical answers (where applicable)

1b(ii) Lower bound is 0.

1b(ii) Maximum is $4/3|\mathcal{S}(\omega)|$ and occurs at $\omega = 1/\sqrt{2}$.

3b d=1.

3d $0 \leq k < 1$.

4a(ii) Not BIBO stable (poles on unit circle).

4b(i) Yes, $Y(t)$ is WSS because $\mathcal{H}$ is LTI and $X(t)$ is WSS.