Version RS/3

EGT3
ENGINEERING TRIPOS PART IIA

Monday 9 May 2022 2 to 3.40

Module 3F2

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 **Assessor comments** Q1. In part (a) many candidates didn't realise that the force through the actuator must be the same as the force through the damper (since they are in series) and that the force through the damper is c times the relative velocity across its terminals. (b)(i) was correctly done by most candidates but in (b)(ii) many forgot how to find the reachability space. In (c) many candidates showed that $v^T A = 0$ and hence v^T is a left eigenvector of A (which was all that was required) though some tried to find the eigenvalue-eigenvector decomposition of A from first principles, which is very lengthy and time-consuming. Most candidates didn't see that $v^T B = 0$ which shows very simply that 0 is an uncontrollable mode. For part (d) many deduced from the state equations that $m_1 \ddot{z}_1 + m_2 \ddot{z}_2$ but too many thought the quantity was energy or momentum.

(a) Newton's second law applied to each mass:

$$\begin{aligned} m_1 \ddot{z}_1 &= c(\dot{w} - \dot{z}_1), & (1) \\ m_2 \ddot{z}_2 &= -c(\dot{w} - \dot{z}_1). & (2) \end{aligned}$$

(Note that the force through the actuator is the same as the force through the damper since they are in series.). Substituting for $\dot{w}$ gives:

$$m_1 \ddot{z}_1 = c(\dot{z}_2 - \dot{z}_1 + u)$$

and similarly for the second equation, which gives the state-space equations as required. [20%]

(b) $m_1 = m_2 = 1$.

(i) Controllability matrix:

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} c & -2c^2 & 4c^3 \\ -c & 2c^2 & -4c^3 \\ 1 & -c & 2c^2 \end{bmatrix}$$

First two rows sum to zero, hence rank deficient and the system is uncontrollable. [20%]

(ii) Second and third columns are parallel, so reachability space is spanned by the first two columns. Normalising these to unit length is one possible answer. Taking linear combinations of the first two columns shows that:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are unit vectors which span the reachability space. [20%]

(c) For general m_1 and m_2 with $v^T = [m_1, m_2, 0]$ we find that $v^T A = 0$, hence v^T is a left eigenvector with eigenvalue 0. Also, $v^T B = 0$ so the system is uncontrollable. [20%]

(d) Taking the sum of (1) and (2) gives

$$m_1 \ddot{z}_1 + m_2 \ddot{z}_2 = 0.$$

Hence $m_1 \dot{z}_1 + m_2 \dot{z}_2$ is constant independent of the actuation, so the velocities of the masses can't be controlled independently. This follows since the coupling between the two masses can only provide an equal and opposite force on the two masses. [20%]

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2 **Assessor comments** This was a popular question which was very well done by many candidates. Part (b)(ii) was carried out equally often by eliminating variables in the simultaneous system equations or by evaluating the expression $C(sI - A)^{-1}B$ by matrix inversion.

(a) (i) At equilibrium:

$$\begin{aligned} 0 &= y_0 - x_0 \\ 0 &= x_0(c - z_0) - y_0 + u_0 \\ 0 &= x_0 y_0 - b z_0 \end{aligned}$$

[15%]

(ii) With $c = 1$:

$$\begin{aligned} y_0 &= x_0 \\ z_0 &= x_0^2/b \\ 0 &= -x_0^3/b + u_0 \end{aligned}$$

which gives the result. [10%]

(b) $b = 4$, $c = 1$ and $u_0 = 2$ gives $y_0 = x_0 = 2$ and $z_0 = 1$.

(i) Substituting $u = u_0 + \delta u$, $x = x_0 + \delta x$ etc and neglecting the quadratic terms gives:

$$\begin{aligned} \dot{\delta x} &= a(\delta y - \delta x) \\ \dot{\delta y} &= \delta x - z_0 \delta x - x_0 \delta z - \delta y + \delta u \\ \dot{\delta z} &= y_0 \delta x + x_0 \delta y - b \delta z \end{aligned}$$

which gives after substituting:

$$\begin{aligned} \dot{\delta x} &= a(\delta y - \delta x) \\ \dot{\delta y} &= -2\delta z - \delta y + \delta u \\ \dot{\delta z} &= 2\delta x + 2\delta y - 4\delta z \end{aligned}$$

[20%]

(ii) Taking Laplace transforms and rearranging gives

$$\begin{aligned} (s + a)\hat{\delta x} &= a\hat{\delta y} \\ (s + 1)\hat{\delta y} &= -2\hat{\delta z} + \hat{\delta u} \\ (s + 4)\hat{\delta z} &= 2\hat{\delta x} + 2\hat{\delta y} \end{aligned}$$

and eliminating $\hat{\delta z}$ gives

$$\begin{aligned}(s+a)\hat{\delta x} &= a\hat{\delta y}\\ (s+1)(s+4)\hat{\delta y} &= -4(\hat{\delta x}+\hat{\delta y})+(s+4)\hat{\delta u}\end{aligned}$$

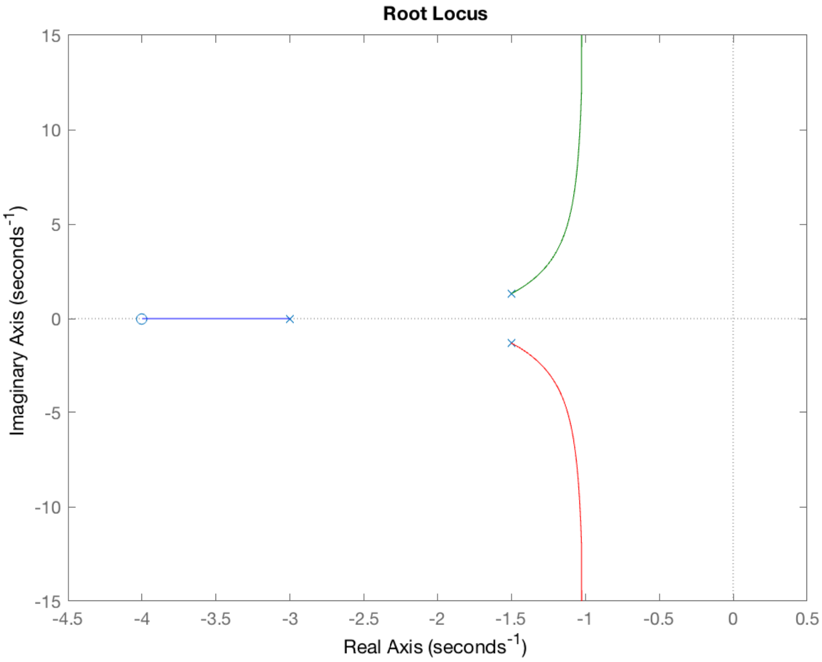
which gives $h(s)\hat{\delta x} = (s+4)\hat{\delta u}$. [20%]

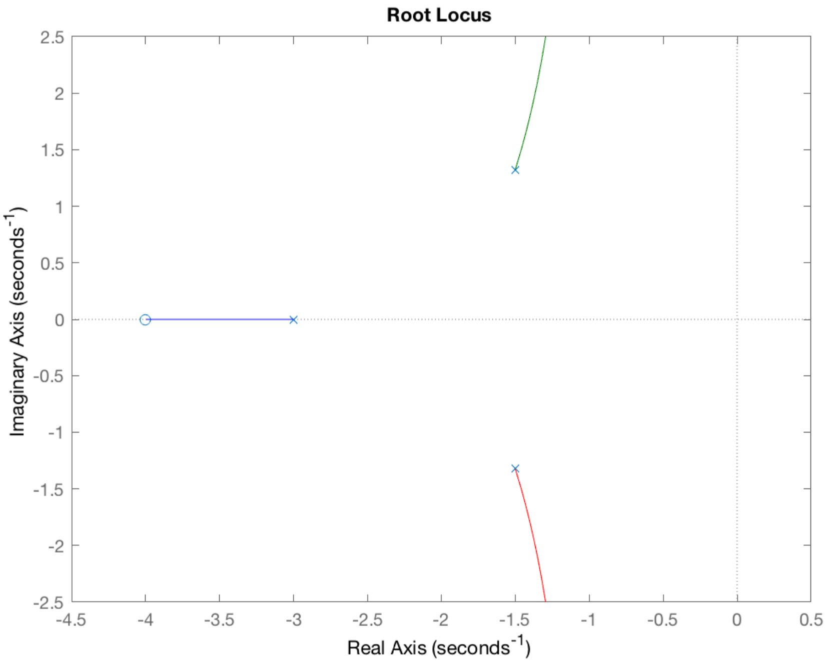
(c) (i) Further with $a = 1$

$$\begin{aligned}h(s) &= s^3+6s^2+13s+12\\ &= (s+3)(s^2+3s+4)\end{aligned}$$

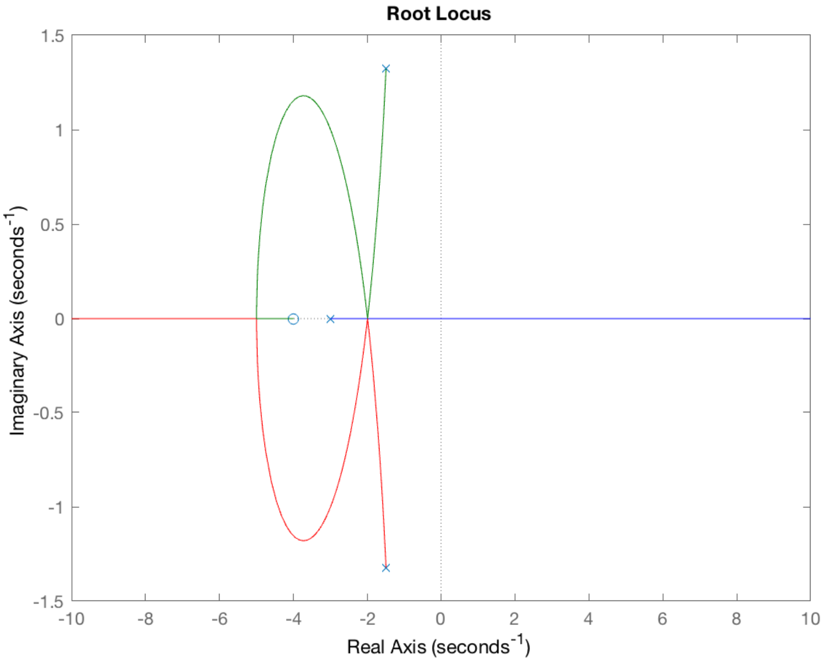
so poles are at: $-3, -3/2 \pm j\sqrt{7}/2 = -1.50 \pm j1.32$. Asymptote centre= -1. Breakaway points are the roots of $s^3+9s^2+24s+20 = (s+5)(s+2)^2$ are not on the real axis part of the root-locus. (There is an interesting triple root coincidence in the root-locus of $-G(s)$. Not asked for in question.) [25%]

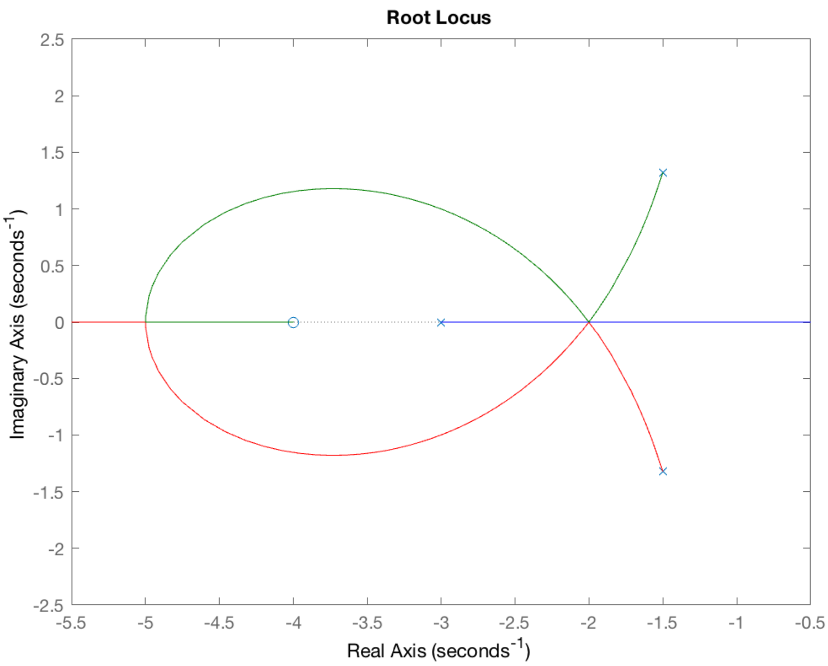
(ii) Near the equiilibrium point the control system is stable but oscillatory. As the proportional feedback gain is increased the frequency of oscillation increases and the damping ratio of these dominant modes decreases. [10%]





Root-locus for $G(s)$





Root-locus for $-G(s)$

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3 **Assessor comments** Q3. Most candidates were able to find the state-space model from the circuit equations (part (a)) and to prove observability (part (c)). Calculation of the transfer function (part (b)) proved more difficult, leading some candidates to check controllability from the rank condition rather than from pole-zero cancellation in part (d). In part (e), only few candidates were able to find a minimal realisation of the uncontrollable circuit, which only amounted to find a state-space representation of the simplified transfer function.

(a) Choosing the state variables $x_1 = v_c$ and $x_2 = i_L$ gives the state-space model

$$\begin{aligned} C\dot{x_1} &= -\frac{2}{R}x_1 + x_2 + \frac{1}{R}u \\ L\dot{x_2} &= -x_1 + u \\ y &= -x_1 + u \end{aligned}$$

[15%]

(b) In the Laplace domain, the state-space model provides the equations

$$\begin{aligned} \hat{x}_2 &= (Cs + \frac{2}{R})\hat{x}_1 - \frac{\hat{u}}{R} \\ Ls\hat{x}_2 &= -\hat{x_1} + \hat{u} \end{aligned}$$

which leads to

$$(LCs^2 + \frac{2Ls}{R} + 1)\hat{x}_1 = (\frac{L}{R}s + 1)\hat{u}$$

and

$$\frac{\hat{y}}{\hat{u}} = \frac{s(s + \frac{1}{RC})}{s^2 + \frac{2}{RC}s + \frac{1}{LC}}$$

[15%]

(c) from the circuit equations, it is easy to express the state variables as functions of the input and output and their derivatives:

$$v_c = u - y,\ i_L = -\frac{y}{R} + \frac{1}{R}(u - y) + C(\dot{u} - \dot{y})$$

Those relationships hold for all (strictly positive) values of the parameters. Hence the system is always observable. [20%]

 (cont.

(d) For $C = L = 1$, the transfer function becomes

$$\frac{\hat{y}}{\hat{u}} = \frac{s(s + \frac{1}{R})}{s^2 + \frac{2}{R}s + 1}$$

Pole zero cancellation occurs in the transfer function if $R = 1$. Because the system is always observable, pole zero cancellation necessarily leads to an uncontrollable model. In this configuration, the transfer function becomes

$$\frac{\hat{y}}{\hat{u}} = \frac{s}{s + 1}$$

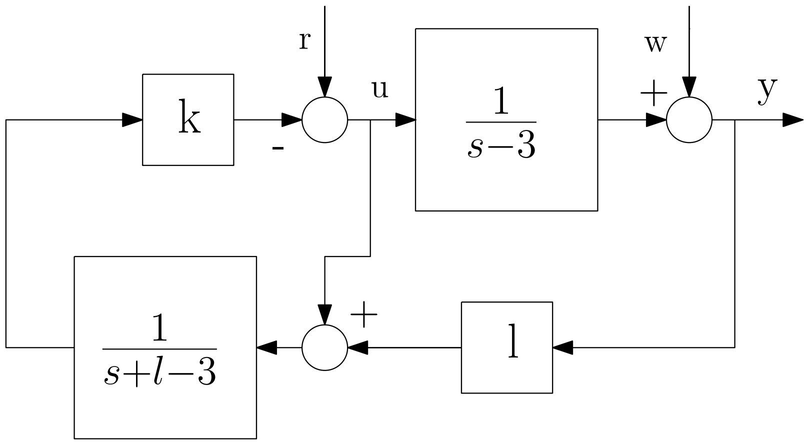
and a state-space realisation is

$$\dot{x} = -x + u, \;\; y = -x + u$$

[20%]

4 **Assessor comments** Q4. The design of the observer (part (a)) and the explanation of the separation principle (part (c)) were done correctly by most candidates. As in Q3, the derivation of the transfer functions in part (b) and (c) proved more difficult. In part (e), only few candidates proposed an open-loop observer, which serves as a filter when the model is trusted more than the measurements.

(a) A state observer is $\dot{\hat{x}} = 3\hat{x} + u + l(y - \hat{y})$, $\hat{y} = \hat{x}$. The error variable $e = y - \hat{y}$ satisfies $\dot{e} = -(l-3)e$. The block diagram of the control system is shown below. [30%]



Block-diagram of feedback control system in Q4.

(b) The two following relations hold:

$$\begin{aligned}\bar{y} &= \bar{w} + \frac{1}{s-3}\bar{u} \\ \bar{u} &= \bar{r} - \frac{k}{s+l-3}(u + ly)\end{aligned}$$

from which one deduces the closed-loop expressions

$$\bar{y} = \frac{s+l+k-3}{s+l-3}\frac{1}{(s+k-3)(s+l-3)}\bar{r} + \frac{1}{(s+k-3)(s+l-3)}\bar{w}$$

[10 %]

(c) Likewise, one deduces

$$\bar{u} = -\frac{kl}{s+l+k-3}\bar{y} + \frac{s+l-3}{s+l+k-3}\bar{r}$$

[10 %]

 (cont.

(d) the separation principle states that the closed-loop poles are the union of the poles of the observer ($3 - l$ in the example) and the state-feedback control system ($3 - k$ in the example). In the limit of $l \to +\infty$ (high-gain observer), the state-feedback control converges towards $u = -ky + r$. which is the desired input if one trusts the output measurement.

(e) A stable plant allows for an open-loop observer ($l = 0$), in which case, the feedback is based on an open-loop model of the output. This is a desirable choice when one trust the model much more than the measurement.

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