1) a) $x_1 = x$, $x_2 = \dot{x}$

$$\Rightarrow \dot{x}_1 = \underbrace{x_2}_{f_1} , \quad \dot{x}_2 = \underbrace{-3x_2 + 3(x_1^2 + x_2^2)x_2 - x_1 + u}_{f_2}$$

b) For equilibrium need $\dot{x}_1 = 0 \Rightarrow x_{2e} = 0$

and $\dot{x}_2 = 0 \Rightarrow 0 = \cancel{-3\times 0} + \cancel{3(x_1^2 + 0^2)}.0 - x_1 + u$

ie $x_{1e} = u_e$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 & -3 + 3x_{1e}^2 \end{bmatrix}$$

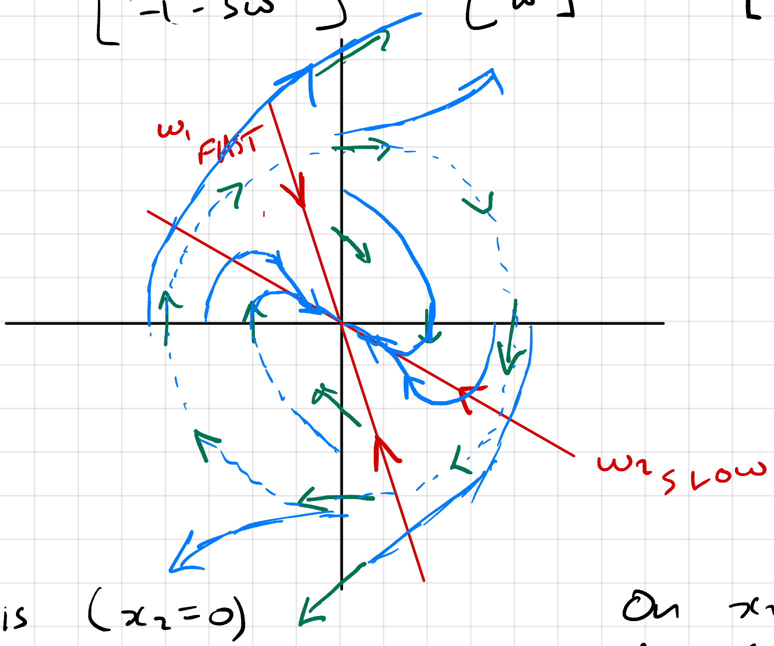
c) $x_{1e} = 0 \Rightarrow \frac{\partial F}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}$

Need eigenvalues & eigenvectors

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ +1 & \lambda+3 \end{vmatrix} = \lambda^2 + 3\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{-3 \pm \sqrt{5}}{2} = \underbrace{-2.618}_{\lambda_1}, \underbrace{-0.382}_{\lambda_2} \Rightarrow$$ origin is Stable

$$A\begin{bmatrix} 1 \\ w \end{bmatrix} = \begin{bmatrix} w \\ -1-3w \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ w \end{bmatrix} \Rightarrow \underline{w}_1 = \begin{bmatrix} 1 \\ -2.62 \end{bmatrix}, \underline{w}_2 = \begin{bmatrix} 1 \\ -0.38 \end{bmatrix}$$



On x_1-axis $(x_2 = 0)$
$\dot{x}_1 = 0$
$\dot{x}_2 = -x_1$

On x_2-axis $(x_1 = 0)$
$\dot{x}_1 = x_2$
$\dot{x}_2 = \underbrace{-3x_2 + 3x_2^3}_{\substack{>0 \text{ if } x_2 > 1 \\ <0 \quad 0 < x_2 < 1}}$

when $x_1^2 + x_2^2 = 1$, $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1$

Along direction of unit circle

The origin is stable, and trajectories do not cross the unit circle. Hence trajectories which start inside unit circle converge to the origin

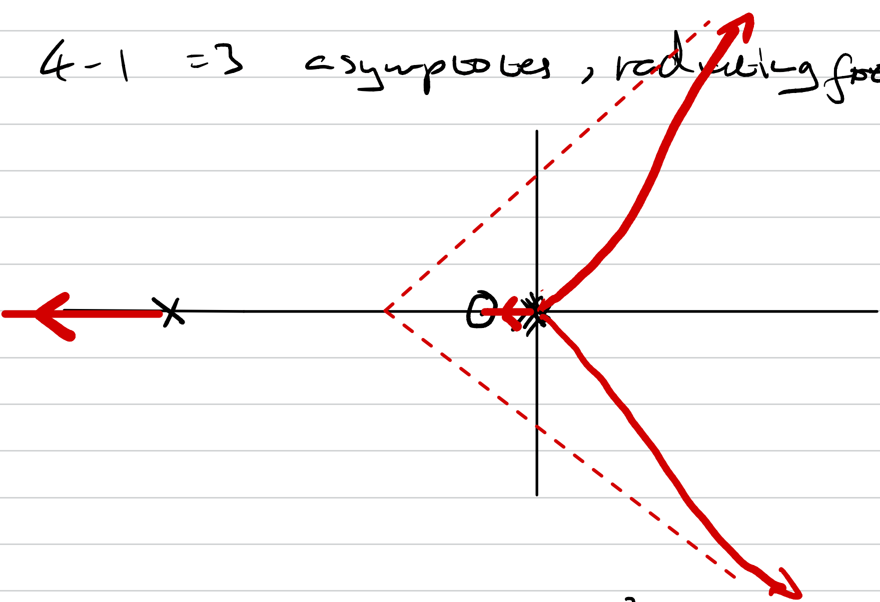
d) It is a useful model that predicts the behaviour of the system accurately when the initial conditions are small and u is small, eg $|x(0)| < 0.1$, $|\dot{x}(0)| < 0.1$, $|u| < 0.1$ (since x_e follows u, and for $x_e \approx 0.1$ the Jacobian $\frac{\partial \underline{f}}{\partial \underline{x}}$ changes very little.

2) a) Need solutions to CLCE : $1 + K G(s) = 0$.

Instead of finding s for fixed K (hard) find s for which solns K are real and positive (angle condition) and then solve for K (gain condition).

b) $G(s) = \dfrac{s+1}{s^3(s+10)}$

$4-1 = 3$ asymptotes, radiating from $\dfrac{0-10+1}{3} = -3$



Breakaway points: $dG/ds = s^3(s+10) - (s+1)(4s^3+30s^2) = 0$

$\Leftrightarrow \quad -3s^4 + 10s^3 - 30s^3 - 4s^3 - 30s^2 = 0$

$\Leftrightarrow \quad s^2(3s^2 + 24s + 30) = 0$

$\Leftrightarrow \quad s^2(s^2 + 8s + 10) = 0$

$\Leftrightarrow \quad s = 0$ (twice), $-4 \pm \sqrt{6} = 0, 0, \underbrace{-1.55, -6.45}_{\text{not on root-locus}}$

CLCE: $K(s+1) + s^3(s+10) = 0 \iff s^4 + 10s^3 + Ks + K$

s^2 coeff $= 0$ for all K, so unstable for all K

$\Rightarrow$ RL does not cross imaginary axis, as shown

ii) $L = K \dfrac{(s+1)^2}{s^3(s+10)^2} = K \dfrac{(-\omega^2 + 2j\omega + 1)}{-\omega^3(-\omega^2 + 20j\omega + 100)}$

$= K \dfrac{(-\omega^2 + 2j\omega + 1)(-\omega^3 \cdot (-\omega^2 - 20j\omega + 100))}{(\text{something real})}$

Need $\operatorname{Re}(\cdot) = 0 \iff \operatorname{Re}\left[(-\omega^2 + 2j\omega + 1)(-\omega^2 - 20j\omega + 100)\right] = 0$

$\iff (1-\omega^2)(100-\omega^2) + 40\omega^2$

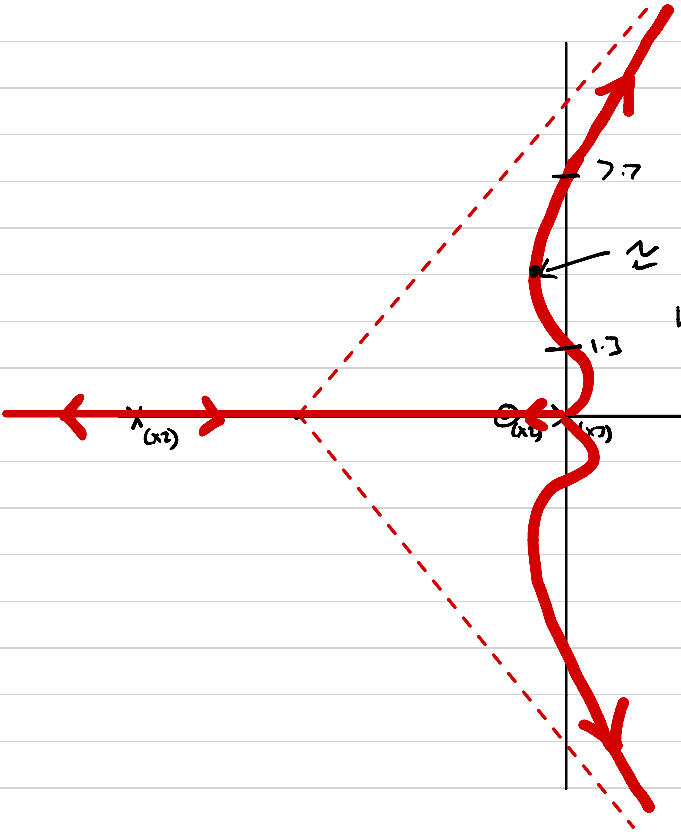
$\iff \omega^4 - 61\omega^2 + 100 = 0$

$\iff \omega^2 = 59.31, 1.69$

$\iff \omega = \pm 7.7, \pm 1.3$

corresponding $K = 1/L(j\omega) = 1207, 83$

$5-2=3$ asymptotes, radiating from $\dfrac{-10 - 10 + 1 + 1}{3} = -6$



$\left(\frac{7.7+1.3}{2}\right)$

$\approx 4.5j$ + small real part

$$k \approx \frac{4.5^3 \times (10^2 + 4.5^2)}{4.5^2} \approx 4.5 \times 120 \approx 540$$

(any reasonable answer, based on an imaginary part between 1.3 and 7.7 and a small real part would be acceptable here)

$\Rightarrow$ stable for $83 < k < 1207$

Rightmost (slowest) poles furthest to the left (fastest) for $k \approx 550$

3 *Assessor comments on Q3: This was an unpopular question. Most candidates who attempted this question solved (a)-(b) and (c), although candidates showed some misunderstanding of controllability at part (b). In part (d), block diagrams were drawn with only minor mistakes, but the transfer function of the controller was typically derived incorrectly from the transfer function of the plant. In part (e), most candidates made a mistake by choosing the third state to be the disturbance. This is likely to be due to mixing up the two examples that are discussed in the lectures: state augmentation for achieving zero steady state and for observing disturbance. In part (f), no candidate derived the correct transfer function and only few candidates recognized that the controller is a classical PID controller. However, many candidates gave good examples for the advantages and disadvantages of state feedback control design.*

A state feedback controller is being designed for position control of a DC motor with transfer function

$$V(s) = \frac{1}{s(s+1)}U(s)$$

(a) Write down a state-space model of the plant.

Sol: A minimal state-space model will have two states as the order of the denominator is two. One good choice for the state is $x_1 = y$ and $x_2 = \dot{y}$. In this case the state space model is given by ($\dot{x} = Ax + Bu, y = Cx$)

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

(b) Explain (preferably without calculation) why your state-space model is both controllable and observable.

Sol: The state-space model is minimal as the following apply: 1) both the dimension of the state and the number of poles are two and 2) there is no pole-zero cancellation in the transfer function. Since a state-space model is minimal if and only if it is both observable and controllable, this system is both observable and controllable.

(c) Design a state feedback controller to place the poles of the closed-loop system at $s = -1$.

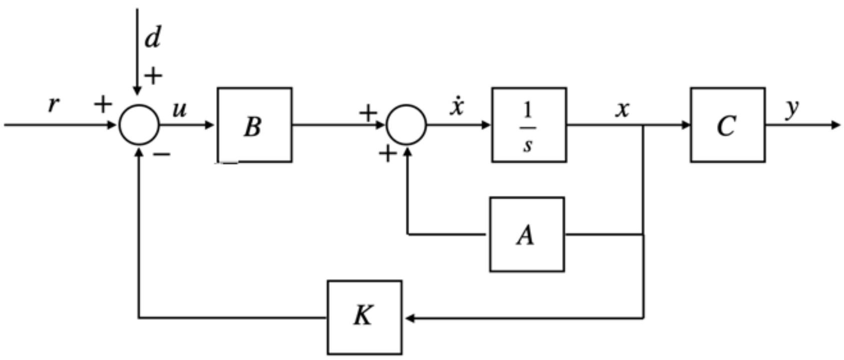
Sol: Since the system is controllable, we can place the poles arbitrarily by a linear state feedback controller in the form of $u = -Kx = -k_1x_1 - k_2x_2$. The elements of the gain matrix K can be calculated by matching the coefficients of the closed loop characteristic polynomial with the coefficients of the target characteristic polynomial $(s+1)^2$. (This method is valid because the poles of a system are the roots of the characteristic polynomial.) The closed loop characteristic polynomial is given by $\det(sI - (A - BK))$, that is

$$\det\left(\begin{bmatrix} s & -1 \\ k_1 & s+1+k_2 \end{bmatrix}\right) = s(s+1+k_2) + k_1$$

After matching the coefficients, we obtain that $k_1 = 1$ and $1 + k_2 = 2 \to k_2 = 1$.

(d) Interpret the state feedback control as the output of a controller with input $-y$ and output u. Calculate the transfer function of this controller and draw a conventional block diagram of the feedback system with a load disturbance signal d added to the input u.

Sol: Putting together that $x_1 = y$ and $x_2 = \dot{y}$, the state feedback control is $u = -Kx = -k_1 y - k_2 \dot{y}$ with transfer function $K(s) = k_1 + k_2 s$, when $-y$ is considered to be the input. A conventional block diagram of this system with input disturbance is shown below.



(e) Assume a constant (but unknown) disturbance $d(t) = d_0$ added at the input. Explain how the state-space model can be augmented by one extra state variable in such a way that any state feedback controller that stabilises the closed-loop system will also ensure zero steady-state error. Can the poles of this feedback system be freely assigned? Briefly justify your answer with no extra calculation.

Sol: Zero steady-state error can be ensured by augmenting our system with a new state $\dot{x}_3 = r - y$. When a state feedback controller stabilizes the system, we get $\dot{x}_3 = r - y = 0$ at steady state, thus we have zero steady state error. By adding this new state, both the linear state feedback controller and the closed loop transfer function have an additional third pole. Therefore, the augmented model with three state space variables is minimal (it can be easily shown that there is no pole-zero cancellation if the controller uses the third state , i.e., if $k_3 > 0$). Minimality infers that the system is controllable and thus the closed loop poles can be assigned arbitrarily.

(f) Determine the transfer function of the augmented controller (with input $-y$) designed in part (e). Do you recognise this as a classical controller structure? Provide both one advantage and one limitation of your state feedback design over classical design methods using Bode and Nyquist plots.

Sol: The linear state feedback controller of the augmented system is in the form of

$$u = -k_1 x_1 - k_2 x_2 - k_3 x_3 = -k_1 y - k_2 \dot{y} - k_3 \int (r - y).$$

Hence, its transfer function with input $-y$ is $K_a(s) = k_1 + k_2 s - \frac{k_3}{s}$, if we assume $r = 0$. This controller is a classical PID controller.

An advantage of state feedback controller is that it can be applied to MIMO systems, whereas graphical tools such as Bode and Nyquist only allow to design controllers of SISO systems. A disadvantage of state feedback control design is that the frequency response of the system can not be designed directly from the state space model. For instance, one would need to calculate and analyze the transfer function to see how the system responds to high frequency noise instead of simply shaping the Bode diagram.

4 *Assessor comments on Q4: This was a popular question that almost all candidates attempted. Parts (a) and (c) were solved well by the majority of the candidates, parts (d) and (e) were often solved with minor mistakes, and parts (b) and (f) proved to be more challenging. The solutions for part (b) were often short, either stating that the matrix Q does not involve B, or that the input can be chosen to be zero, or claiming incorrectly that observability does not involve the input. When part (d) was attempted by first principles, the relationship between x_3 and $\ddot{\theta}$ was often overlooked leading to the wrong conclusion that the system is unobservable. In part (e), candidates tended to focus on the calculation of the gain and often forgot to demonstrate their overall understanding of the observer (e.g., by an equation or a diagram). In part (f), many candidates mentioned that two measurements can improve accuracy but only few of them wrote detailed discussions.*

(a) State a standard test for observability of the linear system

$$\dot{x} = Ax + Bu, \quad y = Cx$$

Sol: A standard test for observability is to check whether the observability matrix is full rank. If $x \in \mathbb{R}^n$ then the observability matrix is defined by $Q = \begin{bmatrix} C^T & (CA)^T & \vdots & (CA^{n-1})^T \end{bmatrix}^T$. Then, the system is observable if $r(Q) = n$. (Note that it is also true that if the system is observable then $r(Q) = n$, but it is not needed for the test.)

(b) Explain why this test does not depend on the matrix B.

Sol: By definition, a system is observable if the state $x(t)$ can be deduced from measurements of the input and the output given at a time period (t_0, t_1), where $t_0 < t < t_1$. Consider the system of differential equations

$$\begin{aligned} y &= Cx \\ \dot{y} &= CAx + f_1(u) \\ &\vdots \\ y^{(n-1)} &= CA^{n-1}x + f_{n-1}(u), \end{aligned}$$

where $\{f_i(u)\}_{i=1}^{n-1}$ are functions of u. When y, its derivatives, and any function of u is given, this system can be solved for x if and only if Q is full rank. Therefore, a system is observable if and only if Q is full rank, where Q does not depend on B.

(c) A cart-mounted inverted pendulum makes an angle θ with the vertical, and the cart moves in a straight line with velocity v. The cart is driven by a force f. For small angles the linearised equations of motion are

$$\begin{aligned} \ddot{\theta} &= \theta + v + f \\ \dot{v} &= \theta - v - f \end{aligned}$$

Define a suitable state vector, keeping the state dimension as small as possible, and write these equations in standard state-space form.

Sol: A suitable state vector that gives a minimal state space model is $x_1 = \theta, x_2 = \dot{\theta}, x_3 = v$. The state space model is then given by the equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u, \quad y = C \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where $u = f$ and C depends on what state is observed.

(d) Is the system observable if only $\dot{\theta}$ (and the force f) are measured? Justify your answer.

Sol: If only $\dot{\theta}$ is observed from the states, then $C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$. Applying the observability test from (a), we first calculate the observability matrix $Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ and then calculate its rank. Since the determinant of the observability matrix is not zero ($|Q| = 2$), it is full rank and the system is observable.

(e) Consider the output $y = \theta$. Design a state observer, locating all the observer poles at $s = -1$.

Sol: If the output $y = \theta$, the output equation is $y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$x. It is easy to see that this system is observable by calculating the rank of the observability matrix or by noticing that $y = x_1$, $\dot{y} = x_2$ and $\ddot{y} - y - u = x_3$. Therefore, we can design a Luenberger observer $\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$, $\hat{y} = C\hat{x}$ with arbitrary poles.

The poles of the observer are the roots of its characteristic polynomial, which is given by (assuming that $L = \begin{bmatrix} l_1 & l_2 & l3 \end{bmatrix}^T$):

$$\det\left(sI - (A - LC)\right) = \det\left(\begin{bmatrix} s + l_1 & -1 & 0 \\ l_2 - 1 & s & -1 \\ l_3 - 1 & 0 & s + 1 \end{bmatrix}\right) = s^3 + (l_1 + 1)s^2 + (l_1 + l_2 - 1)s + (l_2 + l_3 - 2)$$

Matching the coefficients with the coefficients of the target characteristic polynomial $(s + 1)^3$, we obtain that the gain of the observer with poles -1 is $L = \begin{bmatrix} 2 & 2 & 1 \end{bmatrix}^T$.

(f) If measurements of both θ and v are available, discuss why it would be desirable to use both measurements for the estimation of the state, despite the result of part (e).

Sol: First, notice that if measurements of θ are available, then the system is already observable. Therefore, observing v is mainly advantageous when the measurements are noisy or the model is inaccurate.

Based on the state equations, when the observer only relies on θ, it should implement first and second order derivatives of θ to observer all the states. Since a key characteristics of differentiation is to amplify high frequency noise, the observer will inherit this behaviour. If we observe v in addition to θ, then our observer can avoid implementing differentiation and its frequency response can be improved.

One can also notice that with two measured output, there is more freedom in choosing the observer gain (as we have six parameters instead of three). With this freedom, we can choose smaller gain to achieve the same convergence rate. Smaller gain means less injection of measurement noise into the observer which leads to better state estimate.

From another perspective, given the same level of noise amplification, the observer that uses two measurements can apply larger gain compared to the observer that relies on only θ. This is advantageous when there is model uncertainty in addition to measurement noise as higher gain means faster convergence and thus less reliance on the model.