Version GV/4

EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 30 April 2024 9.30 to 11.10

Module 3F2

SYSTEMS AND CONTROL

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) (i) Explain, in one sentence, what is meant by *controllability* of a linear dynamic system. [10%]

(ii) Controllability can be verified by checking that the controllability Grammian is a positive definite matrix. What is the advantage of this criterion with respect to the usual controllability matrix rank condition ? [10%]

(b) Consider the single input single output system

$$\dot{x} = -\Lambda x + fu$$
$$y = f^T x$$

where the matrix Λ is assumed to be diagonal and positive definite.

(i) What are the conditions on the coefficients of the matrix Λ and of the vector f for this system to be controllable ? [30%]

(ii) Show that the impulse response of the system is positive for all $t \geq 0$. Sketch both the impulse response and the step response. [10%]

(iii) For the special case $\Lambda = I$, find a minimal representation of the system. [10%]

(c) The system

$$\dot{x} = -Px + bu$$
$$y = b^T x$$

is called a relaxation system if the matrix P is symmetric, i.e. $P = P^T$, and if it has n distinct positive eigenvalues.

(i) Explain how to determine a change of coordinates $z = Tx$ that transforms any relaxation system in the diagonal form studied in part (b). Find the relationship between the matrices A and Λ and the vectors b and f. Deduce the conditions on the matrix P and on the vector b for a relaxation system to be controllable. [20%]

(ii) Why is such a system called a *relaxation* system ? [10%]

2 (a) Explain the basis for controller design using observers and estimated state feedback as it applies to a state-space system of the form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Mathematical results may be stated without proof. [20%]

(b) A force $u(t)$ is applied to a mass-damper mechanical system with mass M and damping coefficient k. The position $z(t)$ is measured. An observer is to be designed for its velocity.

(i) Write a state-space model for the mechanical model $M\ddot{z}+k\dot{z} = u$ with position and velocity as state variables. [10%]

(ii) Denoting the state of the observer as $\hat{x}$, write down the state-space equation of the observer in terms of its gain matrix H. [20%]

(iii) Find the transfer function of the observer, from u and z to $\hat{x}_1$. If the relationship between z and u satisfies $M\ddot{z}+k\dot{z} = u$, under what further condition will the observer state asymptotically converge to the position and velocity ? [20%]

(iv) Suppose that the position sensor has a constant bias resulting in the measurement $z(t) + b$ for some constant but unknown value b. Explain how the observer designed in part (b)(iii) can be modified to ensure exact estimation of the position (and velocity) despite the sensor bias. [20%]

(v) Justify why the solution in part (b)(iv) will not work in the absence of damping. [10%]

3 Consider the system depicted in Fig. 1 where the measurement is of *either* $x_1(t)$ or $x_2(t)$ and the objective is to be able to control the position of the two masses.

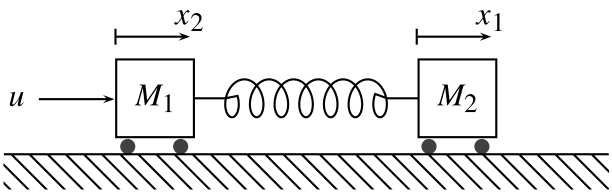

Fig. 1

The two plants have transfer functions

$$P_1: \quad \bar{x}_1(s) = \frac{k}{s^2\left(M_1M_2s^2 + (M_1+M_2)k\right)}\bar{u}(s)$$

and

$$P_2: \quad \bar{x}_2(s) = \frac{M_2s^2 + k}{s^2\left(M_1M_2s^2 + (M_1+M_2)k\right)}\bar{u}(s)$$

where $M_1 = M_2 = 1$ and $k = 2$.

(a) Consider first a feedback controller for P_1 given by $u(t) = K_1\left(r(t) - x_1(t)\right)$. Draw the root-locus diagram for the resulting feedback system, thinking carefully about what parts of the imaginary axis are on the root-locus, and find the smallest value of K_1 for which there is a pole with a real part strictly greater than zero. Does this value make sense physically? [30%]

(b) Now consider a similar controller for P_2: $u(t) = K_2\left(r(t) - x_2(t)\right)$. Again, draw the root-locus and infer that there is no value of K_2 for which there is a pole with a real part strictly greater than zero. [30%]

(c) Now consider a phase-lead compensator for P_2: $\bar{u}(s) = K_2\frac{s+1}{s+2}\left(\bar{r}(s) - \bar{x}_2(s)\right)$. Sketch the form of the resulting root-locus diagram, assuming that there are no break-away points on the real axis between $s = -1$ and $s = -2$, and infer that the feedback system is now stable for all values of K_2. [30%]

Estimate the value of K_2 for which the time constant of the slowest closed loop pole is minimised. [10%]

Hint: In order to differentiate with respect to s you may wish to differentiate first with respect to s^2.

4 Consider the inverted pendulum depicted in Fig. 2 and described by the equations

$$\left.\begin{aligned} mgl\sin\theta &= ml^2\ddot{\theta} + m\ddot{x}l\cos\theta \\ y &= \ddot{x} + z\ddot{\theta}\cos\theta \end{aligned}\right\} \qquad (1)$$

where y is the horizontal acceleration of the point at a distance z along the rod from the pivot.

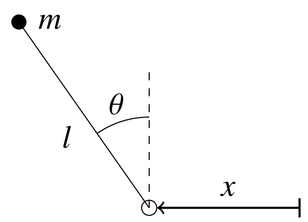

Fig. 2

(a) Put equations (1) into state-space form, with states θ, $\dot{\theta}$, input $u = \ddot{x}$ and output y. [30%]

(b) Linearise your equations about an arbitrary angle θ_e, putting your answer in a standard state-space form with matrices A, B, C and D. [30%]

(c) Find the transfer function from u to y of your linearized model in the standard form as a ratio of two polynomials. [30%]

(d) Without calculation, comment on the observability and controllability of the system as $z \to 0$. [10%]

END OF PAPER

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