Version RS/1

EGT2
ENGINEERING TRIPOS PART IIA

*** May 2024 2 to 3.40

Module 3F2

SYSTEMS AND CONTROL

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed

Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) bookwork

(b) (i) The i-th row of the controllability matrix is of the form

$$[b_i \ -\lambda_i b_i \ \dots \ (-\lambda_i)^{n-1} b_i)] = b_i [1 \ -\lambda_i \ \dots \ (-\lambda_i)^{n-1}]$$

A first condition for full rank of the matrix is that $b_i \neq 0$ for each i. A second condition is that there is no repeated pole, that is, all the λ_i's are different from each other. Under those two conditions, the n rows are independent from each other and the system is controllable.

(ii) The transfer function is $H(s) = c(sI - A)^{-1} b = b^T (s + \Lambda)^{-1} b = \sum_{i=1}^{n} \frac{b_i^2}{s+\lambda_i}$. The impulse response is therefore

$$h(t) = \sum_{i=1}^{n} b_i^2 e^{-\lambda_i t}, \ t \geq 0$$

which is a sum of positive terms. The impulse response is monotonically decreasing and the step response is monotonically increasing, very much like the response of a first-order lag.

(iii) In this case, the transfer function reduces to $\frac{b^T b}{s+1}$. A minimal realisation is given by the first-order system $\dot{z} = -z + \alpha \ u$, $y = \alpha z$, with $\alpha = \| \ b \ \|$. (The rank of the observability matrix is one, hence $n - 1$ states are uncontrollable and the minimal realisation is of dimension one).

(c) (i) A symmetric matrix P with n distinct eigenvalues is diagonisable with an orthogonal change of coordinates, that is, there exists an orthogonal matrix U such that $P = U \Lambda U^T$. The change of variable $z = Ux$ transforms the diagonal system in (b) into $\dot{z} = -Pz + Ubu, \quad y = (Ub)^T x$. Hence a relaxation system is controllable if $b_i \neq 0$ for all i.

(ii) When initialised away from equilibrium with no input, the solution of the system *relaxes* monotonically to the zero equilibrium, without any oscillation.

2 3 4 (a) bookwork.

(b) A force u(t) is applied to a mass-damper mechanical system with mass M and friction coefficient k. The position z(t) is measured. An observer is to be designed for its velocity.

(i) $\dot{x}_1 = x_2$,
$\dot{x}_2 = -\frac{k}{M}x_2 + \frac{1}{M}u$,
$y = x_1$.

(ii) $\dot{\hat{x}}_1 = \hat{x}_2 + h_1(x_1 - \hat{x}_1)$,
$\dot{\hat{x}}_2 = -\frac{k}{M}\hat{x}_2 + \frac{1}{M}u + h_2(x_1 - \hat{x}_1)$.

(iii) The transfer function is

$$\hat{x}_1 = \frac{1}{(s+h_1)(s+\frac{k}{M}) + h_2}\frac{u}{M} + \frac{h_2 + h_1(s+\frac{k}{M})}{(s+h_1)(s+\frac{k}{M}) + h_2}z$$

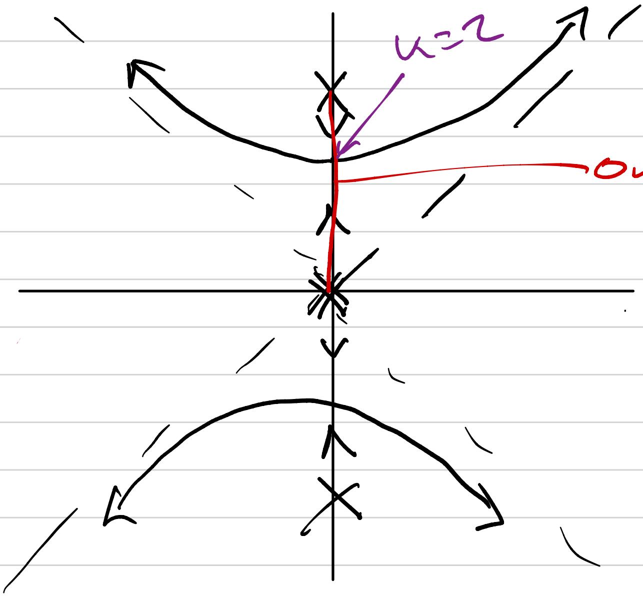
If the relationship $\frac{u}{M} = (s^2 + \frac{k}{M}s)$ holds, then the estimate $\hat{x}_1$ asymptotically converges to the position z provided that the error system is stable. This is ensured if $h_2 > 0$ and $h_1 + \frac{k}{M} > 0$.

(iv) Estimating both the position z and the bias b requires to observe the augmented system
$\dot{x}_1 = x_2$,
$\dot{x}_2 = -\frac{k}{M}x_2 + \frac{1}{M}u$,
$\dot{b} = 0$
$y = x_1 + b$
with the augmented observer $\dot{\hat{x}}_1 = \hat{x}_2 + h_1(y - \hat{y})$,
$\dot{\hat{x}}_2 = -\frac{k}{M}\hat{x}_2 + \frac{1}{M}u + h_2(y - \hat{y})$
$\dot{\hat{b}} = h_3(y - \hat{y})$
$\hat{y} = \hat{x}_1 + \hat{b}$. The gains of the observer must be chosen to ensure stability of the error system.

(v) In the absence of damping, the augmented system is not observable. Indeed, for $u = 0$, one has $y = x_1 + b$, $\dot{y} = x_2$ and $\ddot{y} = 0$, hence the observability matrix has rank 2.

END OF PAPER

3) a) $\frac{K}{s^2(s^2+4)}$ Poles at $0, 0, \pm 2j$



on RL as $\angle$ poles $= -90^\circ + 3 \times 90^\circ$
$= 180^\circ$

$$\frac{d P_2(s)}{ds^2} = \frac{-s^2 - (s^2+4)}{s^2} = -2s^2 - 4$$

$$\Rightarrow \frac{d}{\partial s} P(s) = -2s(2s^2+4) = 0 \quad \text{at } s=0,\ s = \pm\sqrt{2}$$

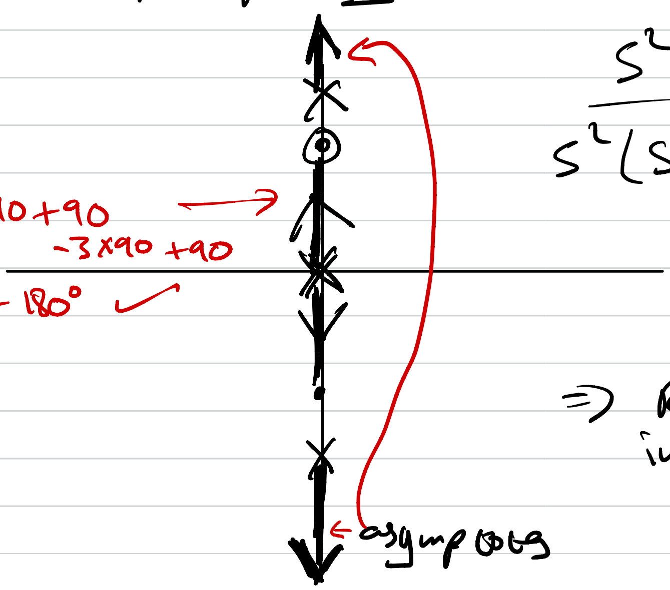
$$s = \sqrt{2}j \Rightarrow 2K_1 = 2.(-2+4) \Rightarrow K_1 = 2$$

Pole in RHP for $K_1 > 2$

Force from a disp of x_1 overpowers restoring force of spring.

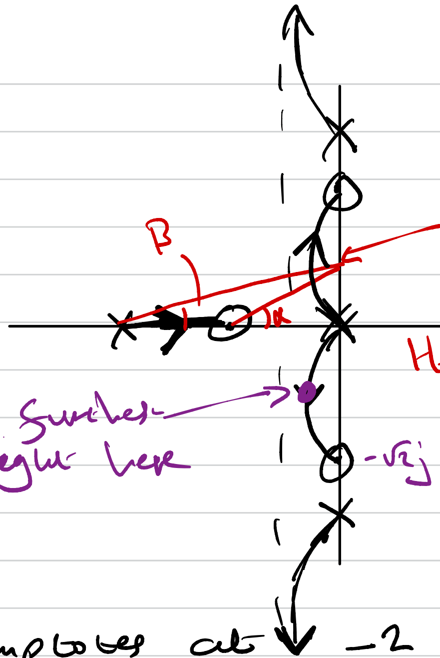
b)

$$\frac{s^2+2}{s^2(s^2+4)}$$



$\angle P_1(j\omega) = -90 + 90 - 3\times 90 + 90 = -180^\circ$

$\Rightarrow$ RL always on imaginary axis c)



On imag axis

$\angle = -90 + 90 - 3\times 90 + 90 + \underbrace{\alpha - \beta}_{\gtrsim 0}$

> -180

Hence extra -ve contribution required from poles & zeros on imaginary axis

poles further to right here

asymptotes at $\frac{-2+1}{5-3} = -\frac{1}{2}$

$2K_2 \approx \frac{\sqrt{2}}{2} \cdot \left(\sqrt{2} + \frac{\sqrt{2}}{2}\right) \Big/ \left(\sqrt{2}^2 \times (2-\sqrt{2}) \times (2+\sqrt{2})\right)$

$\times \sqrt{2} / \sqrt{5} = \frac{2 \cdot \frac{3}{4}}{4} \cdot \sqrt{\frac{2}{5}} \approx \underline{\underline{\frac{1}{4}}}$

4) a) equilibrium:

$$mgl\sin\theta_e = mul\cos\theta_e$$

$$\Rightarrow \tan\theta_e = \frac{u_e}{g} \Rightarrow u_e = g\tan\theta_e$$

$$\ddot{\theta} = \frac{g}{l}\sin\theta - \frac{u}{l}\cos\theta$$

$$x_1 = \dot{\theta},\ x_2 = \theta \Rightarrow \underline{\dot{x}_1 = \frac{g}{l}\sin x_2 - \frac{u}{l}\cos x_2} \leftarrow f_1$$

$$\underline{\dot{x}_2 = x_1}$$

$$y = u + z\dot{x}_1\cos x_2$$

$$= u + z\cos x_2\left(\frac{g}{l}\sin x_2 - \frac{u}{l}\cos x_2\right)$$

$$= u\left(1 - \frac{z}{l}\cos^2 x_2\right) + \frac{zg}{l}\underbrace{\cos x_2\sin x_2}_{\frac{1}{2}\sin 2\theta} \leftarrow g(x_2, u)$$

b) $$\left.\frac{\partial f_1}{\partial x_2}\right|_e = \frac{g}{l}\cos x_{2e} + \frac{u_e}{l}\sin x_{2e}$$

$$= \frac{g}{l}\cos x_{2e} + \frac{g}{l}\frac{\sin x_{2e}}{\cos x_{2e}}\sin x_{2e} = \frac{g}{l\cos x_{2e}}$$

$$\begin{matrix}\delta\dot{x}_1\\ \delta\dot{x}_2\end{matrix} = \overbrace{\begin{bmatrix} 0 & \frac{g}{l\cos x_{2e}} \\ 1 & 0 \end{bmatrix}}^{A}\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} + \overbrace{\begin{bmatrix}\frac{\cos x_{2e}}{l}\\ 0\end{bmatrix}}^{B}$$

$$\left.\frac{\partial g}{\partial x_2}\right|_e = \frac{z}{l}u_e\sin 2\theta_e + \frac{zg}{l}\cos 2\theta_e$$

$$= \frac{z}{l}\cdot\underbrace{g\tan\theta_e\sin 2\theta_e}_{2\sin^2\theta_e} + \frac{g}{l}\cos 2\theta_e = \frac{gz}{l}$$

$$\Rightarrow \delta y = \underbrace{\begin{bmatrix}\frac{gz}{l} & 0\end{bmatrix}}_{C}\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} + \underbrace{\begin{bmatrix}1 - \frac{z}{l}\cos^2 x_{2e}\end{bmatrix}}_{D}u$$

c) $G(s) = C(sI-A)^{-1}B + D$

$$(sI-A)^{-1} = \begin{bmatrix} s & g/l \cos x_{2e} \\ 1 & s \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s & -g/l \cos x_{2e} \\ -1 & s \end{bmatrix}}{s^2 - \frac{g}{l}\cos x_{2e}}$$

$$\Rightarrow G(s) = \frac{\frac{gz}{l} \times s \times \frac{\sin x_{2e}}{l}}{s^2 - \frac{g}{l\cos x_{2e}}} + \left(1 - \frac{z}{l}\cos^2 x_{2e}\right)$$

$$= \frac{\left(1 - \frac{z}{l}\cos^2 x_{2e}\right)s^2 + gz\sin x_{2e}\, s + \left(1 - \frac{z}{l}\cos^2 x_{2e}\right)}{s^2 - \frac{g}{l\cos x_{2e}}}$$

d) $G(s) \to 1$ as $z \to 0$, ie poles at $\pm\sqrt{\frac{g}{l\cos x_{2e}}}$ are cancelled and so realization is not minimal & either loses controllability or observability.
A, B are independent of z, so must lose observability.