EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 6 May 2025   9.30 to 11.10

Module 3F2

LINEAR SYSTEMS AND CONTROL

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A linear system is expressed in state-space form:

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + Bu \\ y &= C\underline{x} \end{aligned}$$

where

$$A = \begin{bmatrix} -3 & 4 & -2 \\ -1 & 2 & 1 \\ 0 & 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

(a) Show that the columns of the matrix W are eigenvectors of A, where

$$W = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad W^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix},$$

and find the corresponding eigenvalues. [10%]

(b) (i) Making use of a suitable state coordinate transformation, or otherwise, find matrices C_1, B_1 and a diagonal matrix Λ so that the impulse response $g(t)$ of the system takes the form $C_1 e^{\Lambda t} B_1$. [10%]

(ii) Hence, or otherwise, find $g(t)$ and note that it takes the form

$$h_1 e^{\alpha_1 t} + h_2 e^{\alpha_2 t}$$

for some constants h_1, h_2, α_1 and α_2. [10%]

(iii) Hence, or otherwise, find the transfer function $G(s)$ of the system. Comment on the relationship between the poles of $G(s)$ and the eigenvalues of A. [15%]

(c) Suppose that $u(t) = 0$ and the state is constrained so that $x_3(t) = 0$. Sketch the state space trajectories in the variables (x_1, x_2) for the system. [25%]

(d) Suppose a feedback law with $u = r - ky$ is applied to the system, where $r(t)$ is a reference signal and k is a constant.

(i) Show that the matrix $A_k = A - kBC$ is the “A-matrix” of the closed loop system. [10%]

(ii) Find the characteristic polynomial of A_k and show that all its roots are coincident when $k = 3/2$. [20%]

2 The swing equation for a synchronous machine takes the form

$$H\ddot{\theta} = P_m - P_e \sin\theta$$

where θ is the rotor angle relative to a reference frame rotating at synchronous speed, P_m is the mechanical power supplied, P_e denotes the maximum electrical power which can be generated, and H is a constant.

(a) With the choice of state variables $x_1 = \theta$, $x_2 = \dot{\theta}$, input $u = P_m$ and output $y = P_e \sin\theta$, write down the nonlinear state-space equations for the system. [10%]

(b) Let the input be held fixed at u_e. What conditions must be placed on u_e so that there is a real equilibrium solution? [10%]

(c) Assuming that the conditions of Part (b) are satisfied show that there are two essentially distinct real equilibrium solutions determined by the following choices of the first state:

$$x_{1,e} = \arcsin\left(\frac{u_e}{P_e}\right) \quad \text{and} \quad x_{1,e} = \pi - \arcsin\left(\frac{u_e}{P_e}\right)$$

[10%]

(d) Find matrices A, B, C and D, with entries which are functions of $x_{1,e}$ and the other constants, for the linearised equations of motion

$$\begin{aligned} \dot{\underline{\delta x}} &= A\underline{\delta x} + B\delta u \\ \delta y &= C\underline{\delta x} + D\delta u \end{aligned}$$

close to an equilibrium point. [25%]

(e) Show that the transfer function from $\delta U(s)$ to $\delta Y(s)$ takes the form

$$G(s) = \frac{a}{s^2 + a} \qquad (1)$$

and note how the sign of a depends on the choice of equilibrium point. Comment on the stability of the equilibrium points. [25%]

(f) Sketch the root-locus diagram for $G(s)$ in (1) with $a = 1$. Suppose a phase lead compensator

$$K(s) = \frac{s+1}{s+5}$$

is selected. Sketch the new root-locus diagram for the compensated system. [20%]

3 The mechanical system shown in Fig. 1 is described by the differential equations

$$m_1\ddot{x}_1 = -k(x_1 - x_2 + x_0) - u$$
$$m_2\ddot{x}_2 = -k(x_2 - x_1 - x_0) + u$$

where x_1 and x_2 denote the positions of the two masses m_1 and m_2, respectively, k is the spring constant, and u is an internal force regarded as a control variable. The constant x_0 is the relative displacement which corresponds to zero spring force.

(a) Write the equations of the mechanical system in the state-space form $\dot{\underline{z}} = A\underline{z} + Bu$, identifying the state vector $\underline{z}$ and the matrices A and B. Pay attention to choose the state vector $\underline{z}$ such that the equilibrium $\underline{z} = \underline{0}$ corresponds to a resting position of the system. [30%]

(b) Show that the system is *not* controllable. Determine the dimensions of the controllable and uncontrollable subspaces. [30%]

(c) Determine an output $y = C\underline{z}$ (with C a nonzero constant vector) whose trajectory is not influenced by the control variable. Hint: use physical intuition to identify a conserved quantity. [20%]

(d) Using the result in Part (c) (or otherwise), determine a new state vector $\tilde{\underline{z}}$ such that the corresponding state-space model is in the controllability canonical form, with matrices $\tilde{A}$ and $\tilde{B}$ having the structure

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix}$$

Calculate those matrices. [20%]

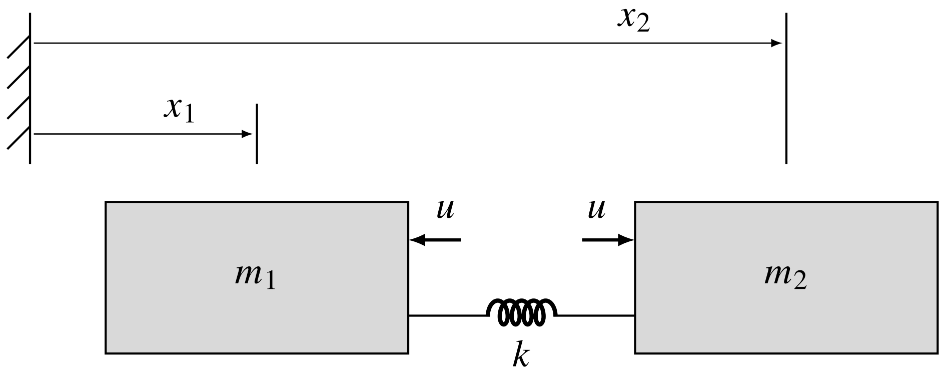

Fig. 1

4 The water reservoir shown in Fig. 2 is modelled via the state-space model

$$\underline{\dot{x}} = \begin{bmatrix} -k_1 & k_1 & 0 \\ k_1 & -(k_1 + k_3) & k_3 \\ 0 & k_3 & -k_3 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} (u - d) \qquad (2)$$

The control variable is the outflow u. The external inflow d is unknown but assumed to be constant.

(a) Explain in a few words the meaning and the design principle of a state observer. [10%]

(b) Assume that the only measured signal is the water level x_2. Under which conditions is the model (2) observable? [20%]

(c) Write the equations of a state observer for a general observer gain $L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$, in the form $\dot{\hat{x}}_i = \ldots, i = 1, 2, 3$. [20%]

(d) Write the error equation of the observer. Prove that the choice $l_1 = l_3 = 0,\ l_2 > 0$ is sufficient to ensure asymptotic convergence of the estimation error. (Hint: you can use either a mathematical or a physical argument to prove your result). [20%]

(e) Explain how the observer design can be modified to estimate the (constant) disturbance d. Write the equations of the modified observer and discuss the choice of the observer gains as in (b), under the same constraint $l_1 = l_3 = 0$. (You are not asked to prove stability of the extended observer). [20%]

(f) The controller $u = -\hat{d}$ is proposed to compensate for the external inflow d. Assuming stability, determine the steady-state behavior of the closed-loop system. [10%]

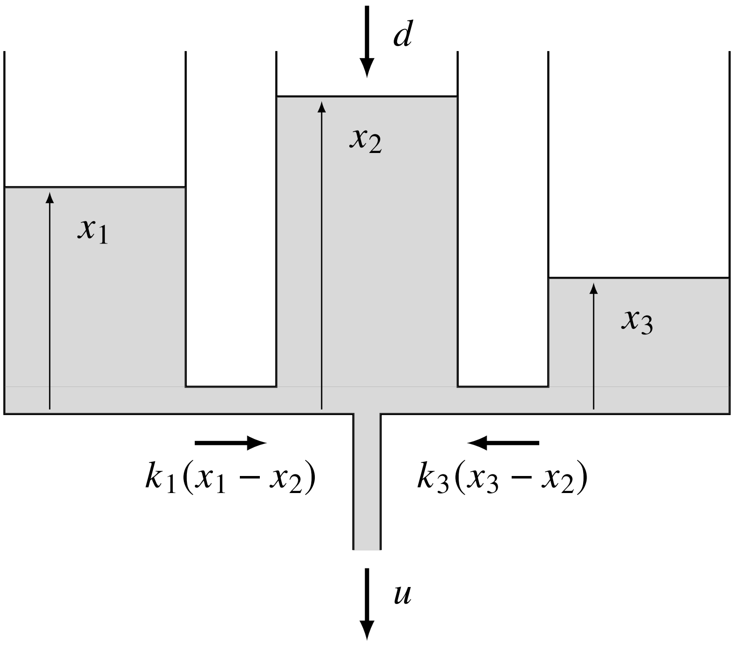

Fig. 2

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