3F2 Solutions 2025

1. (a)

$$A\underline{w}_1 = \begin{bmatrix} -8 \\ -2 \\ 0 \end{bmatrix} = -2\underline{w}_1$$

$$A\underline{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \underline{w}_2$$

$$A\underline{w}_3 = \begin{bmatrix} -5 \\ 0 \\ -5 \end{bmatrix} = -5\underline{w}_3$$

Hence the columns of W are eigenvectors of A with corresponding eigenvalues $-2, 1, -5$. [10%]

(b) (i) System has no D-matrix so $g(t) = Ce^{At}B = CWe^{\Lambda t}W^{-1}B$ where

$$\Lambda = \text{diag}(-2, 1 - 5)$$

$$C_1 = \begin{bmatrix} 4 & 1 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

[10%]

(ii) Hence $g(t) = e^t - e^{-5t}$. [10%]

(iii) Hence

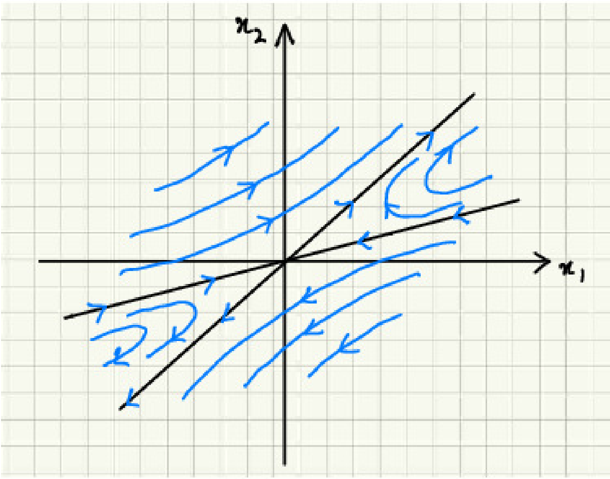
$$G(s) = \frac{6}{(s-1)(s+5)}$$

and we note that the eigenvalue of A at -2 is not a pole of $G(s)$ (and we can see that it is not controllable), and hence the state space realisation is not a minimal realisation of $G(s)$.

[15%]

(c) Can write down from W that the eigenvalues become -2 and 1 with corresponding eigenvectors:

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



[25%]

(d) (i) Substituting for r and y gives:

$$\dot{\underline{x}} = A\underline{x} + B(r - kC\underline{x}) = (A - kBC)\underline{x} + Br$$

which gives the desired result. [10%]

(ii)

$$\begin{aligned} \det(sI - A_k) &= \det \begin{bmatrix} s+3 & -4 & 2 \\ 1+k & s-2 & -1 \\ -k & 0 & s+5 \end{bmatrix} \\ &= (s+5)((s+3)(s-2) + 4(1+k)) - k(4 - 2(s-2)) \\ &= s^3 + 6s^2 + (3+6k)s + -10 + 12k \end{aligned}$$

When $k = 3/2$:

$$\det(sI - A_k) = s^3 + 6s^2 + 12s + 8 = (s+2)^3$$

so all roots are coincident. [This is the only location where they could be coincident since the root at -2 is uncontrollable and can't be moved by feedback.] [20%]

2. (a)

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{P_e}{H}\sin x_1 + \frac{1}{H}u\\ y &= P_e \sin x_1\end{aligned}$$

[10%]

(b) At equilibrium we require

$$u_e = P_e \sin x_{1,e} \tag{1}$$

so for the existence of a real solution we need:

$$\left|\frac{u_e}{P_e}\right| \leq 1$$

[10%]

(c) With the standard meaning of arcsin taking the principal value lying between $-\pi/2$ and $\pi/2$ there are two solutions to (1) in the range 0 to 2π as given. All other solutions of (1) are one of these two plus an arbitrary multiple of 2π, which means there are only two essentially distinct solutions. [10%]

(d)

$$\begin{aligned}\dot{\underline{\delta x}} &= \begin{bmatrix} 0 & 1 \\ \frac{-P_e \cos x_{1,e}}{H} & 0 \end{bmatrix} \underline{\delta x} + \begin{bmatrix} 0 \\ \frac{1}{H} \end{bmatrix} \delta u\\ \delta y &= \begin{bmatrix} P_e \cos x_{1,e} & 0 \end{bmatrix} \underline{\delta x} + 0\delta u\end{aligned}$$

Hence we have:

$$A = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{H} \end{bmatrix}, C = \begin{bmatrix} aH & 0 \end{bmatrix}, D = 0$$

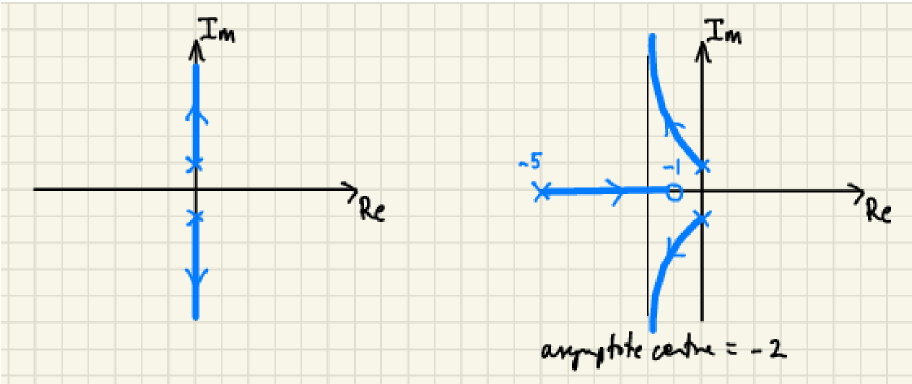
where $a = \frac{P_e \cos x_{1,e}}{H}$. [25%]

(e)

$$\begin{aligned}G(s) &= \begin{bmatrix} aH & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ a & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{H} \end{bmatrix}\\ &= \frac{1}{s^2+a} \begin{bmatrix} aH & 0 \end{bmatrix} \begin{bmatrix} s & 1 \\ -a & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{H} \end{bmatrix}\\ &= \frac{a}{s^2+a}\end{aligned}$$

The transfer function has poles on the imaginary axis for the first equilibrium point when $a > 0$ and so is marginally stable about this equilibrium. The transfer function has poles at $\pm\sqrt{|a|}$ for the second equilibrium point when $a < 0$ and so is unstable about this equilibrium. [25%]

(f)



[20%]

3. (a)

$$z = \begin{bmatrix} x_1 \\ x_2 - x_0 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & 0 & 0 \\ \frac{k}{m_2} & -\frac{k}{m_2} & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{m_1} \\ \frac{1}{m_2} \end{bmatrix}$$

[30%]

(b) The rank of the controllability matrix is 2 because the subspace spanned by A^2B is equal to the subspace spanned by B. Hence both the controllable and uncontrollable subspaces are 2-dimensional. [30%]

(c) The quantity $m_1 z_3 + m_2 z_4$ is constant because $m_1 \dot{z}_3 + m_2 \dot{z}_4 = m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = 0$. The physical interpretation is that the linear momentum of the system is conserved in the absence of external forces. Therefore the output $m_1 z_1 + m_2 z_2$ has a trajectory that is not influenced by the control variable. [20%]

(d) Choose $\tilde{z} = \begin{bmatrix} z_1 \\ \dot{z}_1 \\ m_1 z_1 + m_2 z_2 \\ m_1 \dot{z}_1 + m_2 \dot{z}_2 \end{bmatrix}$. Then $\ddot{\tilde{z}}_3 = 0$, which proves the result. [20%]

4. (a) bookwork. [10%]

(b) The observability matrix has full rank iff $k_1 \neq k_3$. [20%]

(c) The equations of the observer are:

$$\dot{\hat{x}}_1 = k_1(\hat{x}_2 - \hat{x}_1) + l_1(x_2 - \hat{x}_2)$$

$$\dot{\hat{x}}_2 = -k_1(\hat{x}_2 - \hat{x}_1) - k_3(\hat{x}_2 - \hat{x}_3) + l_2(x_2 - \hat{x}_2) - u$$

$$\dot{\hat{x}}_3 = k_3(\hat{x}_2 - \hat{x}_3) + l_3(x_2 - \hat{x}_2)$$

(Note that the disturbance d cannot be included in the observer, as it is an unmeasured signal). [20%]

(d) The error equation is

$$\dot{e} = \begin{bmatrix} -k_1 & k_1 & 0 \\ k_1 & -(k_1 + k_2 + k_3) & k_3 \\ 0 & k_3 & -k_3 \end{bmatrix} e + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} d \qquad (2)$$

which is always stable. The system corresponds to a model of the same reservoir with water levels e_i and with the outflow u replaced by $k_2 e_2$. Alternatively, one can compute the characteristic polynomial and show that the eigenvalues are all negative. [20%]

(e) The modified observer is

$$\dot{\hat{d}} = l_0(x_2 - \hat{x}_2)$$

$$\dot{\hat{x}}_2 = -k_1(\hat{x}_2 - \hat{x}_1) - k_3(\hat{x}_2 - \hat{x}_3) + l_2(x_2 - \hat{x}_2) - (u + \hat{d})$$

with the two remaining equations of the observer unchanged. The new error system corresponds to a model of the same reservoir with no inflow and with the outflow u replaced by $k_2 e_2 + l_0 \int e_2$. This ”reservoir” necessarily converges to an empty state. [20%]

(f) The steady-state behavior is $\hat{d} = d$ (because of the integral action of the controller), and hence $e = 0$. Moreover, the system must be at equilibrium, which imposes $x_1 = x_2 = x_3$. [10%]

Examiner's comments.

Q1.Linear system.

118 attempts, mean 11.62/20, maximum 20, minimum 1.

Solutions to part (a) often didn't take advantage of W being given, so performing a lot of unnecessary calculation. Part (b)(i) and (ii) were sometimes inaccurately done due to incorrect positioning of W and W^{-1} in the state-space transformation. Part (iii) was often done in a lengthy way from scratch, without taking advantage of (b)(ii). Part (c) often didn't use the data already given to find the the eigenvectors for the reduced system. There were lots of mistakes in Part (d) in calculating the characteristic polynomial of A_k.

Q2. Swing equation for a synchronous machine.

150 attempts, mean 13.25/20, maximum 20, minimum 1.

This question was generally very well done by most candidates. The most common failings were in Part (e)—failure to properly link the two equilibrium states to the sign of a—and elementary errors in sketching the root-locus diagram of the compensated system.

Q3. Mechanical system.

100 attempts, mean 12.17/20, maximum 20, minimum 0.

Part (a) and (b) was generally well answered, even though a fairly high number of candidates struggled to define the state in order to have an equilibrium at $z = 0$. About half of the candidates identified that the conserved quantity in (c) was the linear momentum. Part (d) was successfully answered by a smaller number of candidates.

Q4. Water reservoir.

124 attempts, mean 12.64/20, maximum 19, minimum 3.

Most candidates were able to solve the first part of the question (a),(b),(c), even if many candidates wrongly included the (unmeasured) disturbance in the observer. Part (d) often led to lengthy calculations of the characteristic polynomial instead of observing that the error equation had the same structure as the (stable) reservoir equations. About half of the candidates also attempted part (e) and (f), most often (at least partially) successfully.