Version SJG/4

EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 7 May 2025   9.30 to 11.10

Module 3F3

STATISTICAL SIGNAL PROCESSING

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 The vectors $\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \ldots$ follow a first order vector autoregressive process defined by

$$\boldsymbol{X}_{t+1} = A\boldsymbol{X}_t + \boldsymbol{W}_{t+1}$$

where A is a symmetric matrix, $\boldsymbol{W}_t$ are iid multivariate normal random variables with

$$E[\boldsymbol{W}_t] = \boldsymbol{0}$$

$$E\left[\boldsymbol{W}_t \boldsymbol{W}_t^T\right] = I$$

where $\boldsymbol{0} = (0, 0, \ldots, 0)^T$ is the vector of zeros and I is the identity matrix.

(a) Assuming that $\boldsymbol{X}_0 = \boldsymbol{0}$, prove that for $t > 0$

$$\boldsymbol{X}_t = \sum_{s=0}^{t-1} A^s \boldsymbol{W}_{t-s}$$

and thus argue that $\boldsymbol{X}_t$ is multivariate normal. [25%]

(b) Show that $E[\boldsymbol{X}_t] = \boldsymbol{0}$. [10%]

(c) Let $\boldsymbol{Y}_t = A^t \boldsymbol{W}_t$. Show that $E[\boldsymbol{Y}_t \boldsymbol{Y}_t^T] = A^{2t}$. [20%]

(d) Show that $E\left[\boldsymbol{X}_t \boldsymbol{X}_t^T\right] = \sum_{s=0}^{t-1} A^{2s}$. [20%]

(e) Consider the distribution for $\boldsymbol{X} = \lim_{t\to\infty} \boldsymbol{X}_t$.

(i) Show that the covariance of $\boldsymbol{X}$ can be written $\left(I - A^2\right)^{-1}$ (hint: recall the expression for a Geometric Progression).

(ii) What happens if A has any eigenvalues with magnitude larger than 1?

[25%]

2 Let X_t be a discrete random walk. Initially $X_0 = 0$. Then, at each subsequent step, we move to the left or to the right with equal probability, i.e.,

$$X_{t+1} = \begin{cases} X_t + 1 & \text{with probability } 1/2 \\ X_t - 1 & \text{with probability } 1/2 \end{cases}$$

(a) After $2t$ steps, the random walk will be at the origin if the same number of steps to the right have been made as to the left. Hence, show that the probability that $X_{2t} = 0$ is

$$P(X_{2t} = 0) = \frac{(2t)!}{2^{2t}(t!)^2}$$

[20%]

(b) Stirling's approximation for the factorial is

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Use this to show that the probability that $X_{2t} = 0$ is approximately proportional to a power of t, i.e.,

$$P(X_{2t} = 0) \propto t^{-\alpha},$$

and give the value of α. [20%]

(c) Show that the expected number of returns to the origin is infinite. (It will be useful to recall that the series $\sum_{k=1}^{\infty} k^{-\alpha}$ is finite for $\alpha > 1$ and diverges for $\alpha \leq 1$). [20%]

(d) A random walk in 2 dimensions can be modelled as two independent random walks, X_t and Y_t for the x and y coordinates respectively. The random walk has returned to the origin at time $2t$ only if $X_{2t} = Y_{2t} = 0$. Show that in 2 dimensions, the expected number of returns to the origin is also infinite. [20%]

(e) Repeat the calculation for 3 dimensions and comment on the result. [20%]

3 It is desired to estimate a random process X_n formed as an AR process:

$$X_n = \alpha X_{n-1} + 0.2W_n .$$

Observations are made as

$$Y_n = X_n + 0.3V_n$$

where $\{W_n\}$ and $\{V_n\}$ are zero mean, mutually uncorrelated white noise random processes with unity variance.

(a) Determine the autocorrelation function of Y_n and find the range of values of α for which Y_n is wide-sense stationary. [25%]

(b) Under the condition for wide-sense stationarity, show that the power spectrum of $\{X_n\}$ is given by

$$\frac{0.04}{|1 - \alpha \exp(-j\Omega)|^2}$$

[15%]

(c) For α within the stationary range, show that the optimal Wiener Filter for estimating X_n from the observations $, ..., Y_0, ..., Y_n, ...$ has frequency response:

$$H(\exp(j\Omega)) = \frac{A}{(1 - B\cos\Omega)}$$

where A and B are constants that should be determined in terms of α. Sketch the filter frequency response over the range $0 \leq \Omega < 2\pi$ for $\alpha = -0.9$. [25%]

(d) Show that the impulse response of this filter has the form

$$h_n = C\beta^{|n|},\ n = -\infty, ..., +\infty$$

giving equations that express A and B in terms of C and β. [20%]

(e) Can this optimal filter be implemented on-line in a computer programme? If so, then explain how. If not, suggest a suitable alternative strategy for optimal filter design that could be implemented. In either case, there is no need to give a code implementation or the details of the alternative filter design. [15%]

4 (a) It is desired to perform Bayesian estimation $\hat{\theta}$ of a random parameter θ as a function $\hat{\theta}(x)$ of the measured data x, by minimising the expected value of a cost function:

$$C(\hat{\theta}, \theta) \geq 0,$$

which satisfies $C(\theta, \theta) = 0$ for any value of θ.

The likelihood function is $p(x|\theta)$ and a the prior distribution is $p(\theta)$. Show that the optimal choice of estimator $\hat{\theta}$ satisfies the following condition, assuming that $C(\hat{\theta}, \theta)$ is differentiable:

$$\int_\theta \frac{\partial C(\hat{\theta}, \theta)}{\partial \hat{\theta}} p(x|\theta) p(\theta) d\theta = 0$$

where integration is over the full range of possible θ values. [30%]

(b) If minimum mean squared error (MMSE) is required of the estimator, show that the required solution is in the form

$$\hat{\theta} = \frac{1}{Z(x)} \int_\theta \theta p(x|\theta) p(\theta) d\theta$$

and specify $Z(x)$, which does not depend on θ. [20%]

(c) A new cost function is proposed with the form $C(\hat{\theta}, \theta) = 1 - U(\hat{\theta}, \theta)$ where

$$U(\hat{\theta}, \theta) = \begin{cases} 1, & |\hat{\theta} - \theta| < \epsilon \\ 0, & \text{Otherwise} \end{cases}$$

Show that in this case, for the posterior probability $p(\theta|x)$, the optimal estimator $\hat{\theta}$ must satisfy

$$p(\theta = \hat{\theta} + \epsilon|x) = p(\theta = \hat{\theta} - \epsilon|x)$$

and hence explain why, as ϵ becomes small, $\hat{\theta}$ approaches the Maximum *a posteriori* (MAP) estimator θ^{MAP}. [25%]

[Hint: recall that the derivative of an integral is given by $\frac{d \int_0^\phi f(\theta) d\theta}{d\phi} = f(\phi)$ and assume that the probability density functions are smooth and continuous.]

(d) The joint probability density function for two random variables is given by

$$f_{\theta,x}(\theta, x) = \begin{cases} 6\theta, & 0 \leq \theta \leq 1; 0 \leq x \leq 1 - \theta \\ 0, & \text{Otherwise} \end{cases}$$

Find the conditional (posterior) density $p(\theta|x)$ and hence determine the MAP estimator and the MMSE estimator for θ conditional upon data x. Your solution should include a sketch of the non-zero probability region for (θ, x). [25%]

END OF PAPER