## 3F3: STATISTICAL SIGNAL PROCESSING

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**Question 1.** Part (a)

Since $\mathbb{E}\{V_i V_j\} = 0$ when $i \neq j$,

$$\mathbb{E}\left\{X_n^2\right\} = \mathbb{E}\left\{\left(\sum_{j=0}^{t} V_{n-j}^2\right)\right\} + \text{cross-terms} = (t+1)\,\sigma_v^2.$$

Similarly

$$\mathbb{E}\{X_n X_{n+1}\} = t\sigma_v^2$$

because there are $t$ $V_i$-terms in common. Generalising

$$\mathbb{E}\{X_n X_{n+k}\} = (t+1-k)\sigma_v^2$$

for $0 \leq k \leq t$ and is zero for $k > t$.

Part (b)

Check all the conditions for WSS are satisfied. Firstly note that $\sin(2\pi n u) = \sin(2\pi n(u+1/n))$ and thus

$$\mathbb{E}\{\sin(2\pi n U)\} = \int_0^1 \sin(2\pi n u)\,du = 0$$

since sine is integrated over $n$ full periods. $\mathbb{E}\{X_n^2\} < \infty$ and for $n > m$

$$\begin{aligned} 2\mathbb{E}\{X_m X_n\} &= \mathbb{E}\{\sin(2\pi m U)\sin(2\pi n U)\} \\ &= \mathbb{E}\{\cos(2\pi(n-m)U)\} - \mathbb{E}\{\cos(2\pi(n+m)U)\} \end{aligned}$$

which is zero since integrals are computing the area of the cosines over full periods.

Part (c-i)

$\mathbb{E}(X_n) = b_1 + b_2 n + \mathbb{E}(W_n) = b_1 + b_2 n$ so mean is not constant.

$$\begin{aligned} \mathbb{E}\{X_n X_m\} &= (b_1 + b_2 n)(b_1 + b_2 m) + \mathbb{E}\{W_n W_m\} \\ &\quad + \mathbb{E}\{(b_1 + b_2 n) W_m\} + \mathbb{E}\{W_n (b_1 + b_2 m)\} \\ &= (b_1 + b_2 n)(b_1 + b_2 m) + R_W(m-n) \end{aligned}$$

for $m > n$ and $\mathbb{E}\{X_n^2\} = (b_1 + b_2 n)^2 + \sigma_w^2$ where $\sigma_w^2 = \mathbb{E}\{W_n^2\}$.

Thus we also see that $\mathbb{E}\{X_n X_m\} \neq \mathbb{E}\{X_0 X_{m-n}\}$ and so not WSS.

Part (c-ii)

$$Y_n = b_2 + W_n - W_{n-1} = b_2 + \bar{W}_n.$$

$\mathbb{E}(Y_n) = b_2$ so contant mean. $\mathbb{E}\{Y_n^2\} = b_2^2 + \mathbb{E}\left\{\bar{W}_n^2\right\} + 0$ and

$$\mathbb{E}\left\{\bar{W}_n\right\} = \mathbb{E}\left\{W_n^2 + W_{n-1}^2 - 2W_n W_{n-1}\right\} = 2R_W(0) - 2R_W(1)$$

where $R_W(k) = \mathbb{E}\{W_n W_{n+k}\}$. For $k > n$

$$\begin{aligned}\mathbb{E}\{Y_n Y_{n+k}\} =& \mathbb{E}\left\{\left(b_2 + \bar{W}_n\right)\left(b_2 + \bar{W}_{n+k}\right)\right\} \\ =& b_2^2 + \mathbb{E}\left\{\bar{W}_n \bar{W}_{n+k}\right\}\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left\{\bar{W}_n \bar{W}_{n+k}\right\} &= \mathbb{E}\{(W_n - W_{n-1})(W_{n+k} - W_{n+k-1})\} \\ &= R_W(k) + R_W(k) - R_W(k-1) - R_W(k+1).\end{aligned}$$

Thus $Y_n$ has constant mean, finite variance and $\mathbb{E}\{Y_n Y_{n+k}\} = \mathbb{E}\{Y_0 Y_k\}$ and satisfies the conditions for being WSS.

Part (c-iii)

$$\begin{aligned}\mathbb{E}\{\exp(iY_1t_1)\cdots\exp(iY_kt_k)\} &= \exp\left(ib_2(t_1 + \ldots + t_k)\right) \\ &\quad\times \mathbb{E}\{\exp\left(i(W_1 - W_0)t_1\right)\cdots\exp\left(i(W_k - W_{k-1})t_k\right)\}\end{aligned}$$

and using the idependence of $W_k$ we get

$$\begin{aligned}&\mathbb{E}\{\exp\left(i(W_1 - W_0)t_1\right)\cdots\exp\left(i(W_k - W_{k-1})t_k\right)\} \\ &= \mathbb{E}\{\exp\left(i(-W_0)t_1\right)\exp\left(i(t_1 - t_2)W_1\right)\cdots\exp\left(i(t_{k-1} - t_k)W_{k-1}\right)\exp\left(iW_kt_k\right)\} \\ &= \phi(-t_1)\phi(t_1 - t_2)\cdots\phi(t_{k-1} - t_k)\phi(t_k)\end{aligned}$$

The characteristic function of $(Y_1, \ldots, Y_k)$ is thus the same as that of $(Y_m, \ldots, Y_{m+k-1})$ which implies the two random vectors have the same joint probability density function and thus is strictly stationary.

Part (c-iv) Subtract the straight line trend from $X_n$ to get the noise variables only. Now compute the autocorrelation function. Independent random variables have an autocorrelation function that is zero for any non-zero lag. We can see from the figure that the noise variables stay below or above the trend for a length of time which indicates the noise is correlated, i.e. $\mathbb{E}\{W_n W_{n+1}\} > 0$.

**Examiner's comments for Q1:** Parts (a), (b), (c)-(i) were answered well. For part (c)-(ii), the method of the crib avoids lengthy calculations by treating $W_n - W_{n-1}$ as a new random variable. For part (c)-(iii), quite a few wrongly assumed $Y_1, \ldots, Y_k$ were independent when finding the characteristic function. For part (c)-(iv), surprisingly many did not immediately notice that the noise variables must be correlated since they stay below or above the straight line fit in succession.

**Question 2.**
Part (a). To show it is a Markov chain, we need to show

$$\Pr(X_n = i_n | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}) = \Pr(X_n = i_n | X_{n-1} = i_{n-1})$$

which is true as the questions states current bet outcome independent of previous outcomes.

The transition probabilities are $p_{0,0} = 1$, $p_{i,i+1} = \alpha$, $p_{i,i-1} = 1 - \alpha$. All others $p_{i,j}$ (not listed here) are zero.

Sketch of transition diagram is obvious.

[Marks 15%]

Part (b)-i. Let $E_m$ denote the event that the player reaches their target wealth $M$ when starting from $m$. There are many outcomes $o \in E_m$, for example winning every bet until target wealth is reached

$$o = (i_1, \ldots, i_{M-m}) = (m + 1, \ldots, M)$$

or loosing the first and then winning every other one after that,

$$o = (m - 1, m, m + 1, \ldots, M)$$

and so on.

Part (b)-ii. Let $q_m = \Pr(E_m)$, for $m = 0, 1, \ldots, M$. Clearly $q_0 = 0$, $q_M = 1$. For other $q_m$,

$$q_m = \alpha q_{m+1} + (1 - \alpha) q_{m-1}. \tag{0.1}$$

This follows since the result of the first bet is either a win (with probability $\alpha$) or a loss.

For a more advanced explanation: for any outcome $o = (i_1, i_2, \ldots) \in E_m$ either $i_1 = m - 1$ or $i_1 = m + 1$. Thus

$$\Pr(E_m) = \Pr(E_m, i_1 = m + 1) + \Pr(E_m, i_1 = m - 1)$$

and $\Pr(E_m, i_1 = m + 1) = \Pr(i_1 = m + 1) \times \Pr(E_m | i_1 = m + 1) = \alpha q_{m+1}$. [Marks 15%]

Part (c). Rearranging (0.1) gives

$$(1 - \alpha)(q_m - q_{m-1}) = \alpha(q_{m+1} - q_m)$$
$$q_{m+1} - q_m = \frac{1 - \alpha}{\alpha}(q_m - q_{m-1})$$

and iterating gives

$$q_m - q_{m-1} = \left(\frac{1 - \alpha}{\alpha}\right)^{m-1} (q_1 - q_0).$$

[Marks 15%]

Part (d). To solve for $q_1$, first note $q_0 = 0$ and sum the derived relationship to get

$$\begin{aligned} q_M - q_{M-1} + \cdots + q_2 - q_1 &= \left( \left( \frac{1-\alpha}{\alpha} \right)^{M-1} + \cdots + \left( \frac{1-\alpha}{\alpha} \right) \right) q_1 \\ 1 - q_1 &= \left( \left( \frac{1-\alpha}{\alpha} \right)^{M-1} + \cdots + \left( \frac{1-\alpha}{\alpha} \right) \right) q_1 \\ 1 - q_1 &= r(1 + \cdots + r^{M-2}) q_1 \\ q_1 &= \frac{1}{1 + \cdots + r^{M-1}} \end{aligned}$$

[Marks 25%]

Part (e-i).

$$\mathbb{E}\{X\} = \mathbb{E}\{ZU + Y\} = \mathbb{E}\{ZU\} + \mathbb{E}\{Y\} = \mathbb{E}\{Z\}\mathbb{E}\{U\} + \mathbb{E}\{Y\}$$

by linearity and the independence of $Z$ and $U$. $\mathbb{E}\{X\} = \mathbb{E}\{Y\}$.

$$\mathbb{E}\{Z\} = \int_1^2 z \times \frac{3}{7} z^2 dz = \frac{3}{7} \left[ \frac{z^4}{4} \right]_1^2 = \frac{45}{28}.$$

For $\mathbb{E}\{Y\}$ first compute

$$\mathbb{E}\{Y|Z = z\} = 1/z.$$

Thus

$$\mathbb{E}\{Y\} = \int_1^2 \mathbb{E}\{Y|Z = z\} \times \frac{3}{7} z^2 dz = \frac{3}{7} \left[ \frac{z^2}{2} \right]_1^2 = \frac{9}{14}.$$

Now compute the second moments, $\mathbb{E}\{X^2\}$, $\mathbb{E}\{Y^2\}$, $\mathbb{E}\{Z^2\}$.

The term $\mathbb{E}\{X^2\}$ is

$$\mathbb{E}\left\{X^2\right\} = \mathbb{E}\left\{Z^2U^2\right\} + \mathbb{E}\left\{Y^2\right\} = \mathbb{E}\left\{Z^2\right\} + \mathbb{E}\left\{Y^2\right\}$$

since $\mathbb{E}\{ZUY\} = \mathbb{E}\{U\}\mathbb{E}\{ZY\} = 0$, $\mathbb{E}\{U^2\} = 1$ and $\mathbb{E}\{Z^2U^2\} = \mathbb{E}\{Z^2\}\mathbb{E}\{U^2\}$ by independence of $U$ and $Z$.

$\mathbb{E}\{Y^2|Z = z\} = 1/z^2$ thus $\mathbb{E}\{Y^2\} = 3/7$.

$$\mathbb{E}\left\{Z^2\right\} = \int_1^2 z^2 \times \frac{3}{7} z^2 dz = \frac{3}{7} \left[ \frac{z^5}{5} \right]_1^2 = \frac{3}{7} \times \frac{31}{5}.$$

Part (e-ii). Need to calculate

$\mathbb{E}\{XY\}$, $\mathbb{E}\{XZ\}$, $\mathbb{E}\{YZ\}$ and assemble in into

$$\begin{bmatrix} \mathbb{E}\{X^2\} & \mathbb{E}\{XY\} & \mathbb{E}\{XZ\} \\ \mathbb{E}\{XY\} & \mathbb{E}\{Y^2\} & \mathbb{E}\{YZ\} \\ \mathbb{E}\{XY\} & \mathbb{E}\{YZ\} & \mathbb{E}\{Z^2\} \end{bmatrix} - \begin{bmatrix} \frac{9}{14} \\ \frac{9}{14} \\ \frac{45}{28} \end{bmatrix} \begin{bmatrix} \frac{9}{14} & \frac{9}{14} & \frac{45}{28} \end{bmatrix}.$$

$$\mathbb{E}\{YZ\} = \int_1^2 \mathbb{E}\{YZ|Z = z\} \times \frac{3}{7} z^2 dz = \int_1^2 \frac{3}{7} z^2 dz = 1.$$

$$\mathbb{E}\{XY\} = \mathbb{E}\left\{Y^2\right\} = 3/7.$$

$$\mathbb{E}\{XZ\} = \mathbb{E}\{YZ\} = 1.$$

**Examiner's comments for Q2:** Part (a) answered well by most. Part (b)-(i), listing outcomes in $E_m$ was a challenge for some as they did not see an outcome was a sequence of plays that result in attaining target wealth $M$. Part (c) was a challenge for most and many incomplete answers given for part (d), though the steps involved were correct. Part (e) was done well in the main.

**Examiner's comments for Q3:** Part (a): This part answered well in general. Some students forgot the conditions for mean ergodicity. Part (b): Well answered by most. Some very careless solutions. Part (c): Most knew the principle of solution but there were very many slips and inserting of wrong $r_{xy}$ values etc. Part (d): Very few full solutions, most candidates not completing any analysis around the normal pdf

**Examiner's comments for Q4:** Part (a): Very well answered though some careless slips in the differentiation. Part (c): Lots of really good answers to this with full discussion of the three cases. Part (d): Generally quite well done but lots of errors in the integration by substitution. Many very good answers to this [second] part. Candidates clearly familiar with heavy tailed models.

3F3 2021 worked solutions qqs. 3-4

3.(a)

Mean:

$$\mu_n = E[X_n] = -1 \times (1-p) + 1 \times (p)$$

$$= 2p - 1$$

(Constant with time)

$R_{xx}[n, n+m]$

$$= \begin{cases} 1^2 p + (-1)^2(1-p) = 1, & \text{for } m = 0 \\ 1^2 p^2 + (-1)^2(1-p)^2 + 2(1)(-1)\,p(1-p) = (2p-1)^2, & \text{for } m \neq 0 \end{cases}$$

$$= \delta_m\left(1 - (2p-1)^2\right) + (2p-1)^2$$

(Depends only on $m$)

And $R_{xx}[n, n] < \infty$

Hence WSS ✓

3.(a) Contd.

$$C_{xx}[m] = R_{xx}[m] - \mu^2$$

$$= \begin{cases} 1-(2p-1)^2 = -4p^2+4p \\ \qquad = 4p(-p+1), \quad \text{for } m=0, \\ 0, \quad \text{for } m \neq 0. \end{cases}$$

Now, since $C_{xx}[m] \xrightarrow{m \to \infty} 0$

we have mean ergodic.

3(b) With $p = 0.5$,

$R_{XX}[0] = 1$, $R_{XX}[m \neq 0] = 0$,

$R_{XX}[m] = \delta_m$

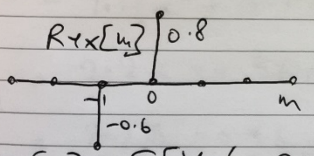
$\{X_n\}$ is WSS, hence $\{Y_n\}$ is WSS.

$R_{YX}[m] = E\left[(0.8X_n - 0.6X_{n-1} + V_n)X_{n+m}\right]$

$= 0.8R_{XX}[m] - 0.6R_{XX}[m+1] + \cancel{E[V_n X_{n+m}]} \to 0$

$= 0.8\,\delta_m - 0.6\,\delta_{m+1}$

Since $E[V_n] = 0$ and $V, X$ independent.




$R_{YY}[m] = E\left[Y_n(0.8X_{n+m} - 0.6X_{n-1+m} + V_{n+m})\right]$

$= 0.8R_{YX}[m] - 0.6R_{YX}[m-1] + E[Y_n V_{n+m}]$

$= 0.8(0.8\,\delta_m - 0.6\,\delta_{m+1})$

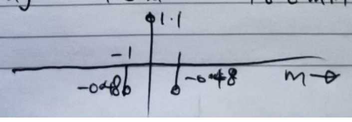
$\quad - 0.6(0.8\,\delta_{m-1} - 0.6\,\delta_m) + R_{YV}[m]$

$= 1\,\delta_m - 0.48\,\delta_{m+1} - 0.48\,\delta_{m-1} + R_{YV}[m]$

Now, $R_{YV}[m] = E\left[(0.8X_n - 0.6X_{n-1} + V_n)V_{n+m}\right]$

$= R_{VV}[m] = \sigma_V^2\,\delta_m$

So $R_{YY}[m] = 1.1\,\delta_m - 0.48\,\delta_{m+1} - 0.48\,\delta_{m-1}$



3.(c)

$$\text{Error} = E\left[(\hat{X}_n - X_n)^2\right]$$

$$= E\left[(\alpha Y_n + \beta Y_{n-1} - X_n)^2\right]$$

$$= \alpha^2 E[Y_n^2] + \beta^2 E[Y_{n-1}^2] - E[X_n^2]$$

$$+ 2\alpha\beta E[Y_n Y_{n-1}] - 2\alpha E[Y_n X_n] - 2\beta E[Y_{n-1} X_n]$$

$$= 1.1\alpha^2 + 1.1\beta^2 - 1 + 2\alpha\beta \times -0.48$$

$$- 2\alpha \times 0.8 - 2\beta \times 0$$

$$= 1.1(\alpha^2 + \beta^2) - 1 - 0.96\alpha\beta - 1.6\alpha$$

$$\frac{\partial}{\partial\alpha} = 2.2\alpha - 0.96\beta - 1.6 = 0 \quad (1)$$

$$\frac{\partial}{\partial\beta} = 2.2\beta - 0.96\alpha = 0 \quad (2)$$

(2) $\alpha = \dfrac{2.2\beta}{0.96}$

(1) $2.2 \times \dfrac{2.2\beta}{0.96} - 0.96\beta = 1.6$

$\beta = 0.392$

$\alpha = 0.8983$

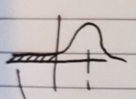
3(d)

[Assume error can be modelled as zero mean Gaussian]

$$MSE = 1.1(\alpha^2 + \beta^2) + 1 - 2\alpha\beta \times 0.48 - 2\alpha \times 0.8$$

$$= 0.5633.$$

Assume we set a decision boundary at 0:
So Probability of error when $x_n = 1$ (or $-1$)



$$= \int_{-\infty}^{-1} N(x|0, 0.5633)\, dx = 1 - \Phi\left(1/\sqrt{0.5633}\right)$$

$$= 0.0914.$$

So error is less than 10%, not bad.

4.

(a)

$$p(y_t|u_t) = \frac{\sqrt{u_t}}{\sqrt{2\pi}} e^{-\frac{u}{2}x^2}$$

Max. likelihood:

$$\frac{d\,p(y_t|u_t)}{du_t} = \frac{1}{\sqrt{2\pi}}\left\{ -\sqrt{u_t}\, e^{-u/2\, x_t^2} \frac{x_t^2}{2} + \frac{e^{-u/2\, x_t^2}}{2} u_t^{-1/2} \right\}$$

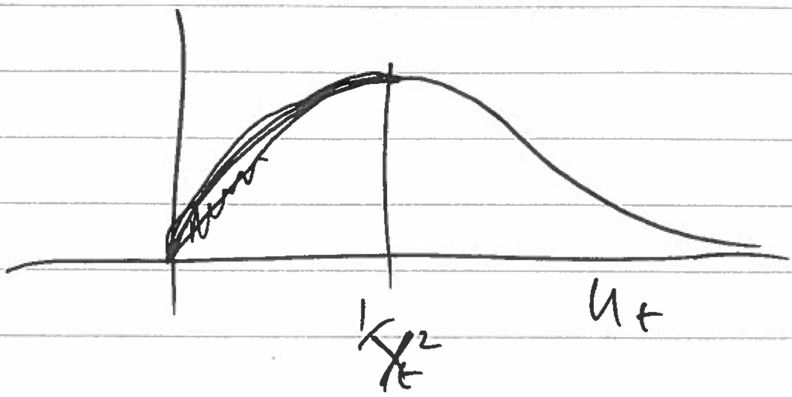
Set to zero:

$$\sqrt{u_t}\,\frac{x_t^2}{2} = \frac{u_t^{-1/2}}{2}$$

Or, $u_t = \dfrac{1}{x_t^2}$ M.L.

4.(a) Contd.

Sketch:




ML Estimate.

(b) This will not be useful in itself as there is very little information about the precision parameter in the observed value.

v



4.

(c) Posterior:

$$p(u_t | y_t) \propto p(y_t | u_t)\, p(u_t)$$

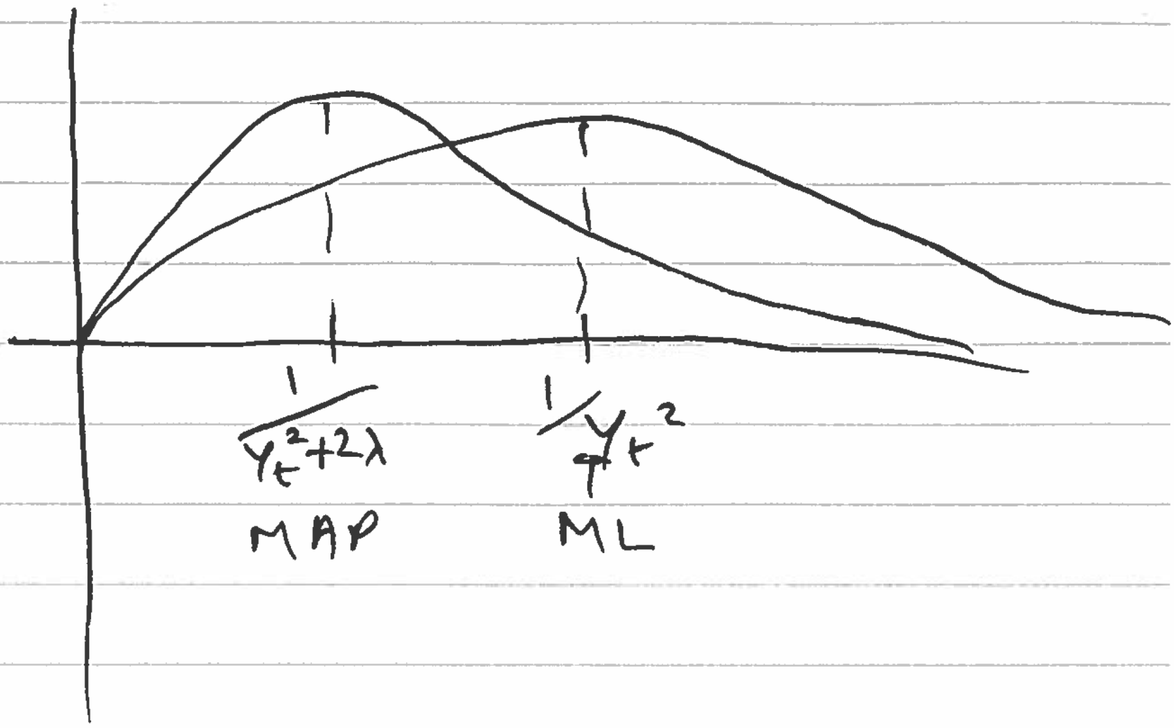
$$= \frac{\sqrt{u_t}}{\sqrt{2\pi}} e^{-y_t^2 u_t / 2} \lambda e^{-\lambda u_t}$$

$$\propto \sqrt{u_t}\, e^{-u_t (y_t^2/2 + \lambda)}$$

The MAP estimate is:

$$u_t^{MAP} = \frac{1}{y_t^2 + 2\lambda}$$

by same calculus as before.




$Y_t$ small rel. $\lambda$:
Estimate is concentrated on $2\lambda$, the prior.

$Y_t$ medium:
Precision is shrunk down toward the prior

$Y_t$ large:
Precision is at the ML estimate.

4.

(d)

$$p(y_t) = \int_0^\infty p(y_t | u_t) p(u_t) \, du_t$$

$$= \int \sqrt{u_t} \, e^{-u_t\left(y_t^2/2 + \lambda\right)} \, du_t$$

Let $V = u_t\left(y_t^2/2 + \lambda\right)$

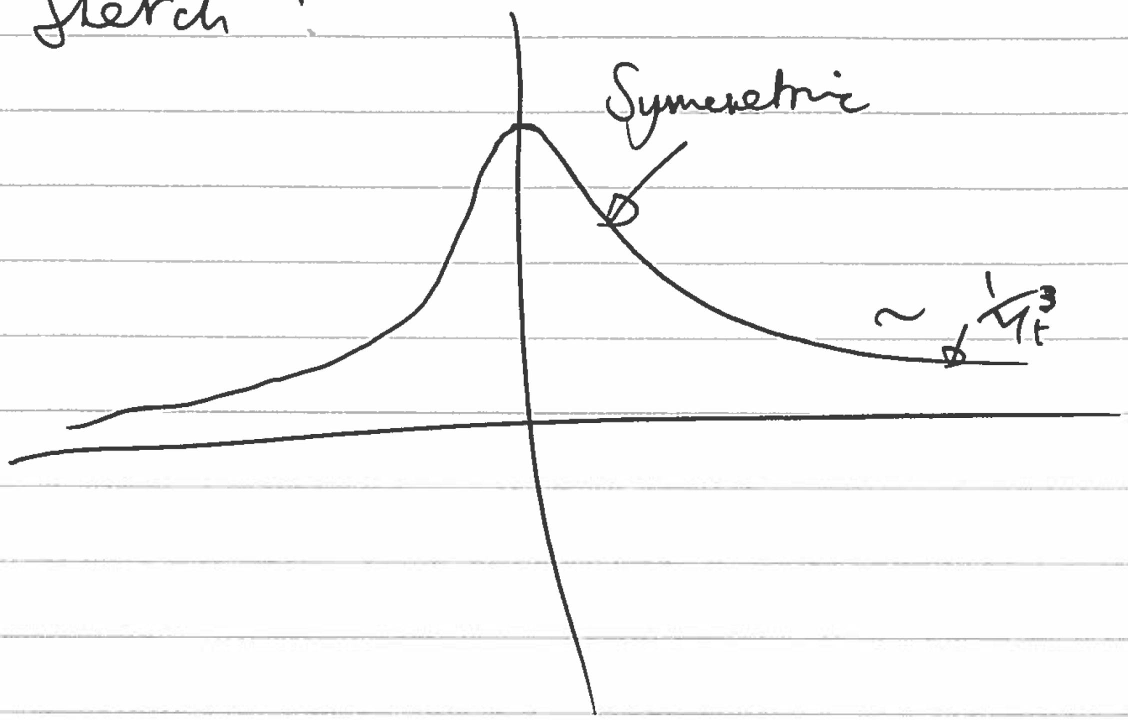
$$\frac{dV}{du_t} = y_t^2/2 + \lambda$$

$$p(y_t) = \int_0^\infty \sqrt{\frac{V}{y_t^2/2 + \lambda}} \, e^{-V} \, \frac{dV}{\left(y_t^2/2 + \lambda\right)}$$

$$= \frac{1}{\left(y_t^2/2 + \lambda\right)^{3/2}} \Gamma\left(3/2\right)$$

(Student's $-t$).

Sketch :




Also a bell-shaped curve, symmetric, but much heavier-tailed than Gaussian. Hence better at modelling extreme values (outliers).

because $1/y_t^3$ decays less rapidly than $e^{-y_t^2/2\sigma^2}$