EGT3
ENGINEERING TRIPOS PART IIA

Wednesday 2nd May 2014 2 to 3.30

Module 3F3

SIGNAL AND PATTERN PROCESSING

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Show that the multiplication of two complex numbers takes 4 real multiplications and 2 real additions. Hence give the *exact* operation count in real multiplies and additions for the length-N DFT. Assume that all complex exponentials have been pre-computed and stored ready for use. [20%]

(b) Discuss the advantages of the FFT over the DFT for computation of the frequency components of a data vector. Give three examples of engineering or scientific areas where the FFT/ DFT are used in practice. [20%]

(c) Explain how the DFT $\mathbf{X} = [X_0\, X_1 \ldots X_{N-1}]^T$ of a length-N signal vector $\mathbf{x} = [x_0\, x_1 \ldots x_{N-1}]^T$ may be written in matrix-vector form as:

$$\mathbf{X} = \mathbf{W}\mathbf{x},$$

giving the form of the matrix $\mathbf{W}$ in terms of the column vectors

$$\mathbf{w}_p = [1\ \exp(-2jp\pi/N)\ \exp(-4jp\pi/N)\ \ldots\ \exp(-2(N-1)jp\pi/N)]^T,\ p = 0, 1, \ldots, N-1.$$

[20%]

(d) Show that

$$\mathbf{w}_q^H \mathbf{w}_p = \begin{cases} N, & q = p, \\ 0, & \text{otherwise}, \end{cases}$$

where p and q may take any values in the range $0, 1, \ldots, N-1$. $\mathbf{w}_q^H$ denotes the conjugate transpose of the column vector $\mathbf{w}_q$.

Hence find the matrix-vector form of the inverse DFT:

$$\mathbf{x} = \mathbf{V}\mathbf{X},$$

where again matrix $\mathbf{V}$ should be expressed in terms of the vectors $\mathbf{w}_p$. [40%]

2 (a) The bilinear transform

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

can be used for the design of a digital filter from an analogue prototype filter.

Show that frequencies map from the analogue to digital domain according to the relationship $\omega = \tan(\Omega/2)$ where ω and Ω should be carefully defined, including their units. [20%]

(b) Discuss how the bilinear transform can be used to design digital filters from analogue prototypes. Your discussion should include a definition of the type of filters that can be designed, stability of the filter, and any distortions that may be introduced to the frequency and impulse responses. [30%]

(c) Explain why the substitution $s = \frac{s'^2 + \omega_l \omega_u}{s'(\omega_u - \omega_l)}$ can be used to take a low-pass analogue prototype and obtain a bandpass filter with lower cutoff at ω_l and upper cutoff at ω_u. [20%]

(d) A digital band-pass filter is required with sample rate of 16kHz. The lower cut-off frequency should be 2kHz and the upper cut-off frequency should be 6kHz. A low-pass analogue prototype filter can be used with transfer function

$$H(s) = \frac{1}{1 + s}.$$

Use this analogue prototype and the bilinear transform to obtain the transfer function of the corresponding band-pass filter and comment on the form of the result. [30%]

3 (a) In some applications it is required to *estimate* a signal observed in noise, and in other cases it is required to *detect* the presence or otherwise of a signal. Explain the circumstances in which a *Wiener* filter could be applied, and those in which a *matched* filter could be the appropriate method, stating the technical assumptions underlying each. Give a practical application of each type of procedure. [30%]

(b) In a neuroscience experiment an auditory stimulus is presented to a student, and the neural response is measured through an electrode taped to the head. The measured electrical response signal may be expressed as a known, *deterministic* signal p_t observed in noise, as follows:

$$x_t = ap_t + e_t,$$

where a is an unknown random variable with zero mean and variance σ_a^2, and $\{e_t\}$ is a zero mean white noise with variance σ_e^2. $\{e_t\}$ is assumed uncorrelated with a for all times t.

Determine the mean value of x_t, $\mu_t = E[x_t]$, and the autocorrelation function $r_{XX}[t_1,t_2] = E[x_{t_1}x_{t_2}]$. Explain with justification whether the process $\{x_t\}$ is wide-sense stationary. [30%]

(c) It is required to estimate the unknown amplitude a when x_t is observed over a fixed time interval $t = 0, 1, ..., T-1$ by filtering the data as follows:

$$\hat{a} = \sum_{t=0}^{T-1} b_t x_t$$

where b_t are the filter coefficients.

The accuracy of estimation of a is to be assessed using the mean-squared error criterion:

$$J = E[(a-\hat{a})^2].$$

Show that J can be expressed as:

$$J = \sigma_a^2 - 2\sigma_a^2 \sum_{t=0}^{T-1} b_t + \sigma_e^2 \sum_{t=0}^{T-1} b_t^2 + \sigma_a^2 \sum_{t_1=0}^{T-1}\sum_{t_2=0}^{T-1} b_{t_1}b_{t_2}p_{t_1}p_{t_2}$$

[40%]

4 Consider a regression problem where the data $\mathscr{D}$ consists of N data points, $\mathscr{D} = \{(x_1, y_1), \ldots, (x_N, y_N)\}$, and you are trying to predict y given x.

(a) Assume a very simple linear regression model:

$$y_n = ax_n + \varepsilon_n$$

where the noise term ε_n is Gaussian with mean zero and variance σ^2. Derive the maximum likelihood (ML) estimates of a and σ^2. [40%]

(b) Now consider a more complex model:

$$y_n = ax_n + bx_n^2 + c + \varepsilon_n$$

with parameters a, b, c and noise variance σ^2 as before. Do you expect the maximum of the likelihood for this model to be lower or higher than for the model in part (a)? Explain your answer. Do you expect the ML estimate of the value of σ^2 to be lower, higher, or the same as in part (a)? Explain your answer. [30%]

(c) Now consider a model like in part (a) but where the noise terms ε_n have a Laplacian distribution,

$$p(\varepsilon_n) = \frac{1}{2\sigma} \exp\left\{ -\frac{1}{\sigma} |\varepsilon_n| \right\}$$

instead of a Gaussian distribution. Explain when using such a Laplacian noise model might be a good idea. [30%]

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