Version SJG/2

EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 1 May 2024   9.30 to 11.10

Module 3F3

STATISTICAL SIGNAL PROCESSING

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 Financial investments can be modelled as random processes. On day n, your investment increases or decreases by a factor of v_n. If you have £Y_n on day n, then on day $n + 1$ you have

$$Y_{n+1} = v_{n+1}\, Y_n.$$

Letting $X_n = \ln Y_n$ and $w_n = \ln v_n$, we can write

$$X_{n+1} = X_n + w_{n+1}.$$

We assume v_i are independently distributed random variables with probability density function

$$p(v_i) = \frac{1}{v_i\sqrt{2\pi}} \exp\left(-\frac{\left(\ln(v_i) + \frac{1}{4}\right)^2}{2} \right),$$

$v_i > 0$. The expected value is $E[v_i] = e^{1/4}$.

(a) Show that $w_i = \ln v_i$ is normally distributed with mean $E[w_i] = -1/4$ and variance $\mathrm{Var}[w_i] = 1$. [30%]

(b) Show that $E[X_n] = -n/4$ and $\mathrm{Var}[X_n] = n$. [10%]

(c) Let $X'_n = X_n + n/4$. Show that X'_n is not WSS. Is X'_n a Markov process? [15%]

(d) Show that $E[Y_n] = e^{n/4} Y_0$. Explain why this appears to be a good investment opportunity. [10%]

(e) For a normally distributed random variable Z with mean μ and variance σ^2 we have the bound

$$P[Z > \mu + \alpha] \le \frac{\sigma}{2\alpha} \exp\left(-\alpha^2/2\sigma^2 \right)$$

for $\alpha > 0$. Using this, find a bound on the probability that X_n is greater than $-n/8$. [20%]

(f) Hence, argue that it is virtually certain that the investment will lose all value as n becomes large. Comment on how this observation fits with the answer to part (d). [15%]

2 A random variable X is Poisson distributed with mean λ if its probability mass function (pmf) is

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

Suppose that $X_1, X_2, X_3, \ldots$ are independent samples from a Poisson distribution with mean λ.

(a) Provide an expression for the joint pmf of X_1 and X_2. [10%]

(b) From this joint pmf, show that

$$P(X_1 + X_2 = n) = \frac{(2\lambda)^n e^{-2\lambda}}{n!}.$$

It may be helpful to use the fact that

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} = 2^n.$$

[25%]

(c) For a random variable X we define the function $G_X(z) = E[z^X]$. Show that for a Poisson random variables X,

$$G_X(z) = e^{\lambda(z-1)}.$$

[20%]

(d) Let k be a Poisson random variable, also with mean λ, and let $S = X_1 + \cdots + X_k$ be a random length sum of the random variables X_i. Show that conditional on k, S is also a Poisson random variable, i.e that,

$$E[z^S|k] = e^{k\lambda(z-1)}.$$

[25%]

(e) Applying the rule of iterated expectation, show that when k is Poisson distributed as above,

$$G_S(z) = E[z^S] = G_X\big(G_X(z)\big).$$

Is the sum S Poisson distributed? Explain your answer [20%]

3 A random process is defined as

$$X_n = \alpha X_{n-1} + W_n$$

where $|\alpha| < 1$ and $\{W_n\}$ is zero-mean white noise with variance σ_W^2.

(a) Show that the autocorrelation function for $\{X_n\}$ satisfies, for $k > 0$,

$$r_{XX}[k] = \alpha r_{XX}[k-1]$$

[15%]

(b) Show that

$$r_{XX}[0] = \frac{\sigma_W^2}{1-\alpha^2}$$

and hence show that

$$r_{XX}[k] = \frac{\sigma_W^2}{1-\alpha^2}\alpha^{|k|}$$

[15%]

(c) Noisy observations of $\{X_n\}$ are made in which

$$Y_n = X_n + V_n$$

where $\{V_n\}$ is zero-mean white noise with variance σ_V^2, independent of $\{X_n\}$. Determine the autocorrelation function of $\{Y_n\}$ and the cross-correlation function between $\{X_n\}$ and $\{Y_n\}$, as a function of α, σ_W^2 and σ_V^2. [20%]

(d) It is desired to predict X_{n+p} where $p > 0$ is a lookahead of p samples, using just the current observed data point Y_n, i.e.

$$\hat{X}_{n+p} = hY_n$$

where h is a coefficient to be determined. Derive the value of h which minimises the mean-squared error of the prediction. [25%]

(e) Derive the mean-squared error for this optimal predictor, and show that as σ_V becomes small the error tends towards:

$$\sigma_W^2 \frac{1-\alpha^{2p}}{1-\alpha^2}.$$

Explain in terms of the model for $\{X_n\}$ why this error tends to zero as σ_W^2 becomes small.

[25%]

4 The exponential random variable X has probability density function

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

with parameter $\lambda > 0$.

(a) Determine from first principles the mean μ and variance σ^2 of the distribution. [15%]

(b) Given N independent samples drawn from the exponential distribution, determine the maximum likelihood (ML) estimator for λ and for μ. [30%]

(c) Is the estimator for μ unbiased? [15%]

(d) What is the variance of the ML estimator for μ? Explain whether the ML estimator is therefore *consistent*. [15%]

(e) A Bayesian prior is to be assigned to λ,

$$p(\lambda) \propto \lambda^{\alpha-1} e^{-\lambda\beta}$$

where $\alpha > 0$ and $\beta > 0$, and the constant of proportionality does not depend on λ. Determine the MAP estimator for λ, and comment on its behaviour as N becomes large. Sketch the posterior probability density function for λ; how would you expect this density to change as N increases? [25%]

END OF PAPER

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