EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 4 May 2022   9.30 to 11.10

Module 3F3

STATISTICAL SIGNAL PROCESSING

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 Two machines, machine A and machine B, both produced 1 kg bags of sugar. Each bag of sugar produced by machine A either weighs 1 kg with probability p or weighs 1.001 kg with probability $1 - p$. A bag of sugar produced by machine B will either weigh 1 kg with probability q or weigh 0.999 kg with probability $1 - q$. Let X_n be the weight of the nth bag of sugar produced by machine A, $n = 1, 2, \ldots$ and let Y_n be the weight of the nth bag of sugar produced by machine B.

A consignment is assembled by taking bags from machines A and B according to the following rule:

$$S_{n+1} = \begin{cases} S_n + X_{n+1} & \text{if } S_n = n \\ S_n + Y_{n+1} & \text{otherwise} \end{cases}$$

where S_n is the total weight of the consignment after selecting n bags. Assume S_0=0.

(a) List the possible values, or range, for the random variables S_1, S_2 and S_n. [10%]

(b) Find $\Pr(S_2 = 2)$ and $\Pr(S_2 = 2.001)$. [10%]

(c) Verify the Markov property $\Pr(S_n = s_n | S_1 = s_1, \ldots, S_{n-1} = s_{n-1}) = \Pr(S_n = s_n | S_{n-1} = s_{n-1})$ is satisfied. [20%]

(d) Find the transition probabilities $\Pr(S_{n+1} = j | S_n = i)$ for all values of i and j in the range of S_n and S_{n+1} respectively. [30%]

(e) Let $\pi_n = \Pr(S_n = n)$. Find π_{n+1} in terms of π_n, p and q. [20%]

(f) For a consignment of K bags, find the expected value of the number of bags that come from machine A. [10%]

2 (a) From measurements, a wide sense stationary random process $\{X_n\}$ is found to have zero mean and its autocorrelation function is estimated to be $\widehat{R}_X(0) = 1, \widehat{R}_X(1) = 0.9, \ldots, \widehat{R}_X(l) = 0.9^l, \ldots$

Find α and the variance of the noise W_n of the first order autoregressive (AR) model

$$X_n = \alpha X_{n-1} + W_n$$

which corresponds to these estimated autocorrelation values and show the corresponding power spectrum estimate is

$$\frac{1 - 0.9^2}{|1 - 0.9 \exp(-j\omega)|^2}.$$

[30%]

(b) Let $\{Y_n\}$ be another (mean zero) first order autoregressive random process, independent from $\{X_n\}$, but with the same estimated autocorrelation function $\widehat{R}_Y(k) = 0.9^k$ for $k \geq 0$. Data from the two autoregressive random processes have been merged together mistakenly to form a single data sequence $Z_0, Z_1, \ldots$

$$Z_n = \begin{cases} X_k & \text{if } n = 2k \\ Y_k & \text{if } n = 2k + 1. \end{cases}$$

For example, $(Z_0, Z_1, Z_2, Z_3, Z_4, Z_5) = (X_0, Y_0, X_1, Y_1, X_2, Y_2)$. Find the autocorrelation function $R_Z(l) = \mathbf{E}(Z_0 Z_l)$ for all l. [30%]

(c) Find the autocorrelation function of the following model

$$Z_n = \alpha Z_{n-2} + \beta_0 W_n + \beta_1 W_{n-1}$$

where $\{W_n\}$ is an independent sequence with zero mean and variance 1. [30%]

(d) Find the values of β_0 and β_1 that best match the autocorrelation found in Part (b). [10%]

3 (a) It is desired to estimate the value of a parameter λ from a sequence of random data. The estimator is denoted $\hat{\lambda}$. Explain the terms *bias* and *variance* of a statistical estimator. The mean-squared error is defined as $MSE = E[(\hat{\lambda} - \lambda)^2]$. Show that

$$MSE = variance + bias^2$$

and hence explain why it is not always best to design an unbiased estimator. [20%]

(b) In a low illumination imaging experiment it is assumed that the number of photons N_k arriving at a particular pixel in time interval k is independent and Poisson distributed with mean $\mu_k = a \exp(-k)$, $k = 0, ..., K-1$, i.e.

$$p(N_k|a) = \frac{\mu_k^{N_k} e^{-\mu_k}}{N_k!}$$

It is proposed to estimate the underlying constant a using the following formula:

$$\hat{a} = \alpha \sum_{k=0}^{K-1} w_k N_k$$

where w_k are positive weights such that $\sum_{k=0}^{K-1} w_k = 1$, and where α is a constant. Determine the value of α which makes the bias of the estimator zero, for a fixed set of weights w_k. [20%]

(c) Show that the MSE of the estimator, under the unbiased condition derived above, can be written as:

$$a \frac{\sum_{k=0}^{K-1} w_k^2 e^{-k}}{(\sum_{k=0}^{K-1} w_k e^{-k})^2}$$

[30%]

(d) Determine and sketch the likelihood function for a, clearly marking any salient points. Determine the Maximum likelihood (ML) estimator and discuss how it relates to the weighted estimator above. What is the bias and variance of the ML estimator? [30%]

4 (a) A random process is defined as:

$$x_n = A\cos(n\omega_0 + \phi) + B$$

where $A \sim \mathcal{N}(\mu_A, \sigma_A^2)$ and $B \sim \mathcal{N}(0, \sigma_B^2)$ are mutually independent Gaussian random variables. ω_0 is a fixed frequency and ϕ is a random variable uniformly distributed between 0 and π, independent from A and B.

Show that the autocorrelation function of $\{x_n\}$ is

$$R_{xx}[n_1, n_2] = \sigma_B^2 + 0.5(\mu_A^2 + \sigma_A^2)\cos((n_2 - n_1)\omega_0)$$

and determine whether the process is wide-sense stationary (WSS). [25%]

(b) A WSS process has zero mean and autocorrelation function

$$R_{xx}[n_1, n_2] = \cos((n_2 - n_1)\omega_0)$$

where ω_0 is a constant. Determine whether the process is mean-ergodic.

[25%]

(c) The WSS process in Part (b) above is observed in independent white Gaussian noise:

$$y_n = x_n + v_n, \quad v_n \sim \mathcal{N}(0, \sigma_v^2)$$

It is desired to predict the next value x_{n+1} based on two previous observed values, y_{n-1} and y_n, using a 2-tap FIR filter:

$$\hat{x}_{n+1} = h_0 y_n + h_1 y_{n-1}$$

Derive, in terms of ω_0, the filter coefficients h_0 and h_1 that minimise the mean-squared error (MSE) in this prediction.

[30%]

(d) Show that as the noise becomes very small (i.e. $\sigma_v^2 \to 0$) the optimal coefficients are:

$$h_0 = 2\cos\omega_0, \quad h_1 = -1$$

Calculate the MSE for this optimal filter and comment on this behaviour. [20%]

END OF PAPER

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