Solutions: 3F4 2015

ENGINEERING TRIPOS   PART IIA

Friday 1st May 2015   2 to 3.30

Module 3F4

DATA TRANSMISSION

Version: NGK/3

1 (a) Equalisers reduce the effect of intersymbol interference (ISI) and so improve bit error rate (BER) performance.

(i) The zero forcing equaliser aims to completely eliminate ISI at the sampling instants. To achieve this, the equaliser response must be

$$H_E(z) = \frac{1}{P(z)}$$

where $P(z)$ is the z-transform of the sampled received pulse response $p(n)$.

(ii) The ZF equaliser can give rise to unwanted noise amplification if the channel frequency response has nulls at particular frequencies. In this case the ZF equaliser will have high gain at these frequencies. To overcome this problem, the MMSE equaliser aims to minimise the error between the received symbols and the transmitted symbols, thus explicitly including the effects of noise in the design process. Thus a compromise between ISI reduction and noise enhancement is achieved.

(iii) In a DFE, delayed and weighted *detected symbols* are fed back and used to cancel ISI. Since detected symbols are used, they are noise-free and so noise enhancement does not occur. However problems do occur if an error is made in a received symbol. In this case, a burst of errors is likely to occur at the equaliser output.

[30%]

(b)

$$p(n) = 1,\ 0.9,\ 0,\ 0,\ \ldots$$
$$\therefore\ P(z) = 1 + 0.9z^{-1}$$

So the ideal ZF equaliser is

$$H_E(z) = \frac{1}{1+0.9z^{-1}}$$

which is a first-order IIR filter with feedback coefficient -0.9. [10%]

(c) To get a 4-coefficient FIR approximation to the IIR ZF equaliser, we need the first 4 terms of the binomial expansion of $H_E(z)$, which can be obtained either by polynomial division or by the binomial theorem:

$$H_E(z) = (1+0.9z^{-1})^{-1} = 1 - 0.9z^{-1} + 0.81z^{-2} - 0.729z^{-3} + \ldots$$

Hence

$$H_{FIR}(z) = 1 - 0.9z^{-1} + 0.81z^{-2} - 0.729z^{-3}$$

and the FIR-equalised pulse response is

$$P_{FIR}(z) = P(z)\, H_{FIR}(z) = (1 + 0.9z^{-1})\,(1 - 0.9z^{-1} + 0.81z^{-2} - 0.729z^{-3})$$

$$= 1 - 0.6561z^{-4}$$

For the specified unipolar scheme:

With no equaliser:

Worst case '1' $= 1.0$ v

Worst case '0' $= 0 + 0.9 = 0.9$ v

Worst case eye opening, $h = 1 - 0.9 = 0.1$ v

$$\therefore \text{ Ratio } \frac{h}{\sigma} = \frac{0.1}{\sigma}$$

With equaliser:

Worst case '1' $= 1.0 - 0.6561 = 0.3439$ v

Worst case '0' $= 0$ v

Worst case eye opening, $h_e = 0.3439 - 0 = 0.3439$ v

Noise variance at equaliser output,

$$\sigma_e = \sigma\sqrt{1^2 + 0.9^2 + 0.81^2 + 0.729^2} = \sigma\sqrt{2.9975} = 1.73\sigma$$

$$\therefore \text{ Ratio } \frac{h_e}{\sigma_e} = \frac{0.3439}{1.73\sigma} = \frac{0.2}{\sigma} \quad \text{(twice as good as before)}$$

[40%]

(TURN OVER for continuation of SOLUTION 1

(d) For the given MMSE equaliser:

$$H_{FIR}(z) = 1 - 0.8z^{-1} + 0.5z^{-2} - 0.3z^{-3}$$

$$\therefore P_{FIR}(z) = P(z)\,H_{FIR}(z) = (1 + 0.9z^{-1})\,(1 - 0.8z^{-1} + 0.5z^{-2} - 0.3z^{-3})$$
$$= 1 + 0.1z^{-1} - 0.22z^{-2} + 0.15z^{-3} - 0.27z^{-4}$$

With equaliser:

Worst case '1' $= 1.0 - 0.22 - 0.27 = 0.51$ v

Worst case '0' $= 0 + 0.1 + 0.15 = 0.25$ v

Worst case eye opening, $h_e = 0.51 - 0.25 = 0.26$ v

Noise variance at equaliser output,

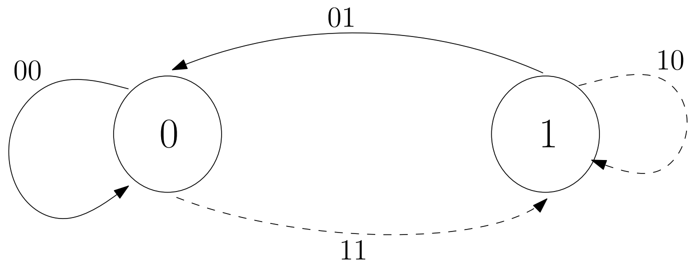
$$\sigma_e = \sigma\sqrt{1^2 + 0.8^2 + 0.5^2 + 0.3^2} = \sigma\sqrt{1.98} = 1.407\sigma$$

$$\therefore \text{ Ratio } \frac{h_e}{\sigma_e} = \frac{0.26}{1.407\sigma} = \frac{0.185}{\sigma}$$

We can see that the worst case eye opening is worse for the MMSE equaliser (0.26, compared with 0.34 for the ZF one). However the rms noise for the MMSE is lower (1.41σ, compared with 1.73σ). Since the worst case eye opening occurs relatively infrequently (all 4 of the ISI bits have to be the worst case polarity), we find in practice that the lower effective noise of the MMSE equaliser improves the *average bit error rate* performance compared with the ZF equaliser.

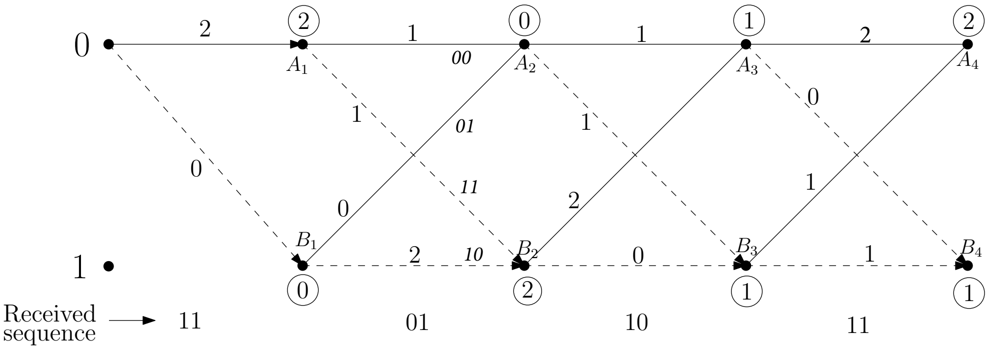
[20%]

2 (a) (i) The state diagram of the code is shown below. It has only 2 states because the state does not need to include the left-most bit (the input bit) in the shift register. Transitions due to input bit 0 are shown in solid lines, and those due to input 1 are shown in dashed lines. The edges are labeled with the code bits corresponding to the transitions.



[15%]

(ii) We use the Viterbi algorithm. The trellis representation of the code is shown in the Figure below.



- The 2 state values are shown at the left. The four pairs of bits in italics show the 2 code bits produced by each edge into nodes A_2 and B_2. These apply into A_n and B_n for any state transition time, n.
- The numbers on the edges indicate the distance of the output of the transition from the corresponding bits of the received sequence (which are shown at the bottom of the trellis).
- The numbers that are encircled at the nodes indicate the *minimum distance* of the corresponding node from the origin (node 0).

We see that the minimum distance path is $0 - B_1 - A_2 - A_3 - B_4$, with a total distance from the received sequence of $0+0+1+0 = 1$ bit.

The decoded sequence is 11 01 00 11.

The corresponding input sequence is 1 0 0 1. [40%]

(b) (i) The error pattern has length $n = 5$, and the syndrome has length $(n-k) = 3$. Hence $k = 2$, and the rate is $2/5$. [5%]

(ii) The parity check matrix H has dimension $(n-k) \times n$, i.e., 3×5. The syndrome $\underline{s}$ for an error pattern $\underline{e}$ is $\underline{e}H^T$, i.e., it is of the form

$$\underline{s} = [s_1\ s_2\ s_3] \ = \ \underline{e}H^T \ = \ [e_1\ e_2\ e_3\ e_4\ e_5] \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Notice that the syndrome for the error pattern [0 0 0 0 1] is the last row of H^T; similarly the syndrome for the error pattern [0 0 0 1 0] is the fourth row of H^T etc. Therefore, reading off the corresponding syndromes from the table, we conclude that H^T has the following form:

$$H^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The parity check matrix is thus

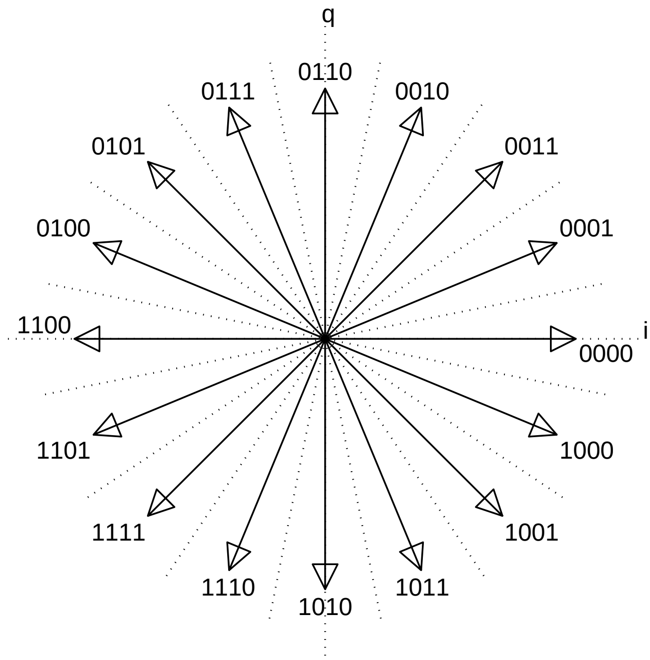
$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

[25%]

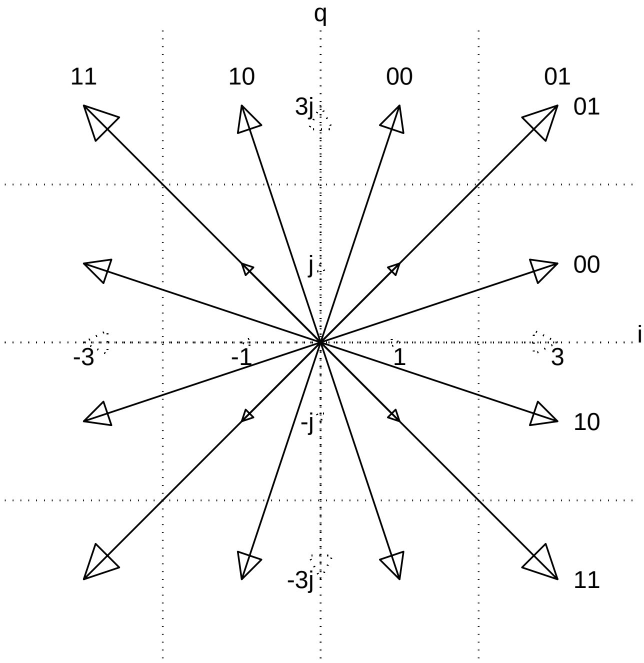
(iii) Observing that no two columns of H sum to $\underline{0}$, but the sum of the second, third, and fourth columns is $\underline{0}$, we conclude that $d_{min} = 3$. This code can therefore correct $\frac{d_{min}-1}{2} = 1$ error and detect $d_{min} - 1 = 2$ errors. [15%]

3 (a) The phasor constellations are:

16-PSK:



16-QAM:



[20%]

 (TURN OVER for continuation of SOLUTION 3

(b) M-PSK constellations are constrained to be constant amplitude. Hence they lie on a circle in the phasor diagram and the distance between points will decrease linearly $\propto \frac{1}{M}$ with increasing M. The points of a QAM constellation are spread uniformly in 2-D space and hence the distance between points decreases only with $\frac{1}{\sqrt{M}}$.

There is only a small difference in the spacings when $M = 16$, but the difference of spacing between PSK and QAM doubles for every 4-fold increase in M. [20%]

(c)

$$s(t) = \mathrm{Re}[p(t)\ e^{j\omega_c t}] = \frac{1}{2}\left[p(t)\ e^{j\omega_c t} + p^*(t)\ e^{-j\omega_c t}\right]$$

If

$$p(t) \Leftrightarrow P(\omega) = \int_{-\infty}^{\infty} p(t)\ e^{-j\omega t} dt$$

then

$$p^*(t) \Leftrightarrow P^*(-\omega) = \left[\int_{-\infty}^{\infty} p(t)\ e^{j\omega t} dt\right]^*$$

and

$$p(t)\ e^{j\omega_c t} \Leftrightarrow P(\omega - \omega_c)$$

The Fourier transform of $p^*(t)\ e^{-j\omega_c t}$ is then given by

$$\int_{-\infty}^{\infty} p^*(t)\ e^{-j\omega_c t}\ e^{-j\omega t} dt = \int_{-\infty}^{\infty} p^*(t)\ e^{-j(\omega+\omega_c)t} dt = P^*(-(\omega+\omega_c))$$

Substituting into the FT of the equation for $s(t)$ gives

$$S(\omega) = \frac{1}{2}\left[P(\omega - \omega_c) + P^*(-(\omega+\omega_c))\right]$$

[25%]

(d) For both M-PSK and M-QAM, the symbols are rectangular pulses of duration T_s, where

$$T_s = \frac{1}{\text{symbol rate}} = m\ .\ \frac{1}{\text{bit rate}} = \frac{m}{R}$$

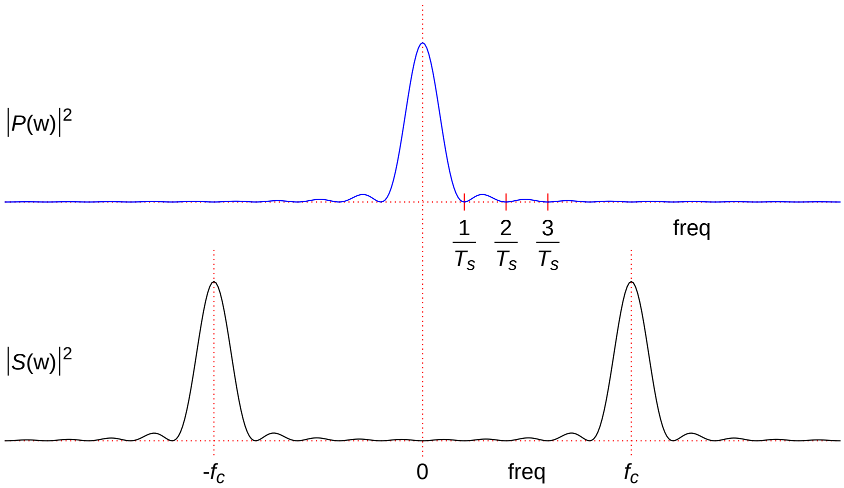
The pulses are zero mean and uncorrelated with each other (assuming random binary data), so the phasor spectrum $P(\omega)$ will be proportional to $\mathrm{sinc}\left(\frac{\omega T_s}{2}\right)$, the spectrum of te rectangular pulse $g(t)$.

With random symbols, the power spectrum $|P(\omega)|^2$ will be proportional to $|\ \mathrm{sinc}\left(\frac{\omega T_s}{2}\right)|^2$. The first zeros of $|P(\omega)|^2$ will therefore occur when $\frac{\omega T_s}{2} = \pm\pi$, i.e. when $\omega = \pm\frac{2\pi}{T_s}$ rad s^{-1} or $f = \pm\frac{1}{T_s}$ Hz.

When modulated onto a carrier of frequency ω_c rad s^{-1} or f_c Hz, the power spectrum is given by:

$$|S(\omega)|^2 = \frac{1}{4}\left[|P(\omega-\omega_c)|^2 + |P(-(\omega+\omega_c))|^2\right]$$

as shown below, assuming negligible overlap of the shifted spectral components centred on f_c and $-f_c$.



Hence the RF bandwidth will be from $(f_c - \frac{1}{T_s})$ to $(f_c + \frac{1}{T_s})$ Hz

$$= \frac{2}{T_s} = \frac{2R}{m} = \frac{2R}{\log_2 M} \ \text{Hz}$$

This applies to both M-PSK and M-QAM. [35%]

4 (a) The 2 key advantages of digital methods over analogue for broadcast signals are:

- The ability of digital signalling to reject channel noise at every stage of the transmission channel;
- Greater capacity of digital transmissions when sophisticated source compression methods are used, and when single-frequency transmission is feasible from a network of transmitters.

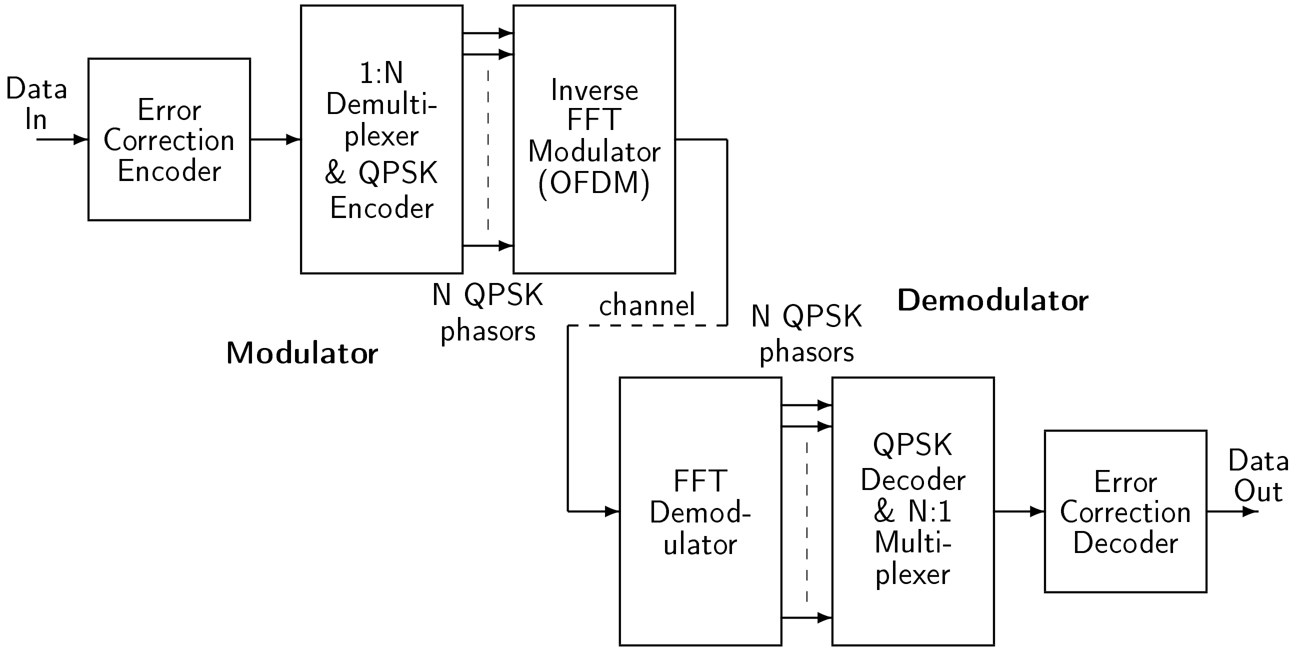
The main disadvantage of digital formats is their substantially increased demodulation and decoding complexity, but this can now be provided cheaply by VLSI chips. The availability of complex chips has made feasible source compression and complicated signal coding and modulation formats, and thus has driven the move to digital in order to provide many more channels within the limited spectrum bandwidth available.

[15%]

(b) Digital audio is designed to be received in cars and other moving vehicles, so there is a need for a highly robust modulation format that has optimal noise-rejection properties and is resilient to rapidly changing multi-path channels. Hence, for audio, QPSK was chosen as a highly robust modulation with fairly good spectral efficiency (two-times better than BPSK).

Digital video must handle the much higher data rates of video signals, compared with audio, and so needs a highly spectrally efficient modulation. Therefore 64-QAM was chosen as being 3 times better in bandwidth efficiency than QPSK (6 bits / symbol, instead of just 2 bits / symbol). Televisions are normally in fixed locations (not in vehicles) so fixed directionally-selective antennae can be used. This reduces the multipath delay spread and largely eliminates the problems of rapidly time-varying channels. Hence the lower noise tolerance of 64-QAM is acceptable for video broadcasts. [20%]

(c) The block diagram for a COFDM system is shown below (from the lecture notes):



The key feature of the DFT is the *orthogonality* of the DFT subcarriers (tones). As long as they are spaced $\frac{1}{T_D}$ Hz apart in frequency, then each subcarrier can be demodulated independently of the others, without any mutual interference, where T_D is the DFT analysis period or block length. It is important that no data-induced phase or amplitude transitions occur within the DFT analysis period – otherwise orthogonality is destroyed and there will be ISI between subcarriers. The FFT is a computationally efficient way of implementing the DFT, especially when $N = 2^n$. [20%]

(d) Multi-path delays tend to cause frequency-selective fading, which 'knocks out' a few of the DFT subcarriers in a random way. Thus ECC is needed to allow the source data to be recovered, despite the losses of some data over the channel.

Guard periods (bands) are needed between each DFT block, so that data-induced phase transitions can be kept within the guard periods and thus the transitions will not affect the orthogonality of the DFT subcarriers during the analysis periods. [20%]

(e) If the subcarrier spacing is 1.5 kHz, in order to preserve orthogonality the analysis period of the DFT must be

$$\frac{1}{1500} = 0.667 \text{ ms}$$

Adding the guard period of 0.1 ms means that the block transmoission rate will be

$$\frac{10^3}{(0.667+0.1)} = 1304 \text{ symbol s}^{-1} \text{ on each subcarrier.}$$

With 1200 subcarriers and 2 bit / symbol (for QPSK), the channel data rate will be

$$1200 \times 2 \times 1304 = 3.13 \,.\, 10^6 \text{ bit s}^{-1}$$

This is the rate available for the redundantly encoded data from the ECC encoder. Hence the user data rate will be half of this $= 1.565 \,.\, 10^6$ bit s^{-1}.

The bandwidth of the 1200 subcarriers will be approx

$$1200 \times 1.5 \text{ kHz} = 1.8 \text{ MHz}$$

In practice some additional multiples of 1.5 kHz would need to be added between COFDM signals to allow for adequate decay of the $\operatorname{sinc}(x)$ sidelobes of the outer subcarriers – say 50 to 100 kHz at each end of the block of 1200 subcarriers, increasing the total bandwidth to 1.9 or 2.0 MHz. It is also usual to leave one subcarrier slot empty at the centre of the band, to provide a 'key' for frequency offset correction in the receiver.

[25%]

END OF SOLUTIONS