3F4 Data Transmission 2022
Crib

Question 1

Consider an M-ary modulation scheme where for $m = 1, \dots, M$, the m-th waveform is given by

$$s_m(t) = \begin{cases} 0 & \frac{(m-1)T}{M} \leq t \leq \frac{mT}{M} \\ +A & \text{otherwise} \end{cases}$$

(a) Calculate the energy of each of the signals $s_1(t), \dots, s_M(t)$ and the average energy of the signal constellation E_s.

All signals have the same energy, and thus the average energy is equal to the energy of each of the individual signal energies. The energy of a signal is equal to

$$E_\mathrm{s} = A^2 T \frac{M-1}{M}.$$

(b) Specify an orthonormal basis for the signal set, specify the dimension and give the vector representation of each signal.

An orthonormal basis is

$$\phi_m(t) = \begin{cases} 0 & \frac{(m-1)T}{M} \leq t \leq \frac{mT}{M} \\ \sqrt{\frac{M}{T(M-1)}} & \text{otherwise} \end{cases}$$

for $m = 1, \dots, M$. The dimension is M. The vector representation is

$$\boldsymbol{s}_1 = (A\sqrt{\frac{T(M-1)}{M}}, 0, \dots, 0) = (\sqrt{E_\mathrm{s}}, 0, \dots, 0) \tag{1}$$

$$\boldsymbol{s}_2 = (0, A\sqrt{\frac{T(M-1)}{M}}, 0, \dots, 0) = (0, \sqrt{E_\mathrm{s}}, 0, \dots, 0) \tag{2}$$

$$\vdots \tag{3}$$

$$\boldsymbol{s}_M = (0, \dots, 0, A\sqrt{\frac{T(M-1)}{M}}) = (0, \dots, 0, \sqrt{E_\mathrm{s}}). \tag{4}$$

(c) We consider transmission over an additive Gaussian noise channel with power spectral density $\frac{N_0}{2}$.

(i) Assuming that symbols are equiprobable, derive the optimal detector.

This is bookwork from the lectures.

$$P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{s}_m) = P_{Y|X}(y_1|s_{m,1} = 0) \cdots P_{Y|X}(y_m|s_{m,m} = \sqrt{E_\mathrm{s}}) \cdots P_{Y|X}(y_M|s_{m,M} = 0) \tag{5}$$

$$= \frac{1}{(\sqrt{\pi N_0})^M} e^{-\frac{y_1^2}{N_0}} \cdots e^{-\frac{(y_m - \sqrt{E_\mathrm{s}})^2}{N_0}} \cdots e^{-\frac{y_M^2}{N_0}} \tag{6}$$

$$= \frac{1}{(\sqrt{\pi N_0})^M} e^{-\frac{\sum_{i=1}^M y_i^2}{N_0}} e^{-\frac{E_\mathrm{s}}{N_0}} e^{\frac{2\sqrt{E_\mathrm{s}} y_m}{N_0}} \tag{7}$$

Assuming equiprobable symbols, the maximum likelihood is the optimal detector. It computes the following estimate

$$\hat{m} = \arg\max_{m=1\dots,M} P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x} = \boldsymbol{s}_m) \tag{8}$$

$$= \arg\max_{m=1\dots,M} e^{\frac{2\sqrt{E_\mathrm{s}}y_m}{N_0}} \tag{9}$$

$$= \arg\max_{m=1\dots,M} y_m, \tag{10}$$

i.e., the optimal detector outputs the position m of the largest component of the received vector.

(ii) Assume that $M = 2$. Show that the pairwise probability of error is given by

$$p_e = Q\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right).$$

and compare it with the error probability of 2-PAM.
With $M = 2$ we have that the error probability is

$$p_e = \frac{1}{2}p_e(\boldsymbol{s}_1 \text{ was transmitted}) + \frac{1}{2}p_e(\boldsymbol{s}_2 \text{ was transmitted}). \tag{11}$$

The first term gives

$$p_e(\boldsymbol{s}_1 \text{ was transmitted}) = \mathbb{P}[y_2 \geq y_1] \tag{12}$$

$$= \mathbb{P}[n_2 \geq \sqrt{E_\mathrm{s}} + n_1] \tag{13}$$

$$= \mathbb{P}[n_2 - n_1 \geq \sqrt{E_\mathrm{s}}]. \tag{14}$$

Since $n_2 - n_1$ is Gaussian with zero mean and variance N_0, we have that

$$p_e(\boldsymbol{s}_1 \text{ was transmitted}) = Q\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right).$$

The second term $p_e(\boldsymbol{s}_1$ was transmitted) is identical. Thus, the overall error probability is

$$p_e = Q\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right).$$

Since the error probability of 2-PAM is

$$p_e = Q\left(\sqrt{2\frac{E_\mathrm{s}}{N_0}}\right),$$

2-PAM gives a better error probability thanks to the factor 2 and the fact that the Q-function is decreasing. Indeed, it takes twice the energy to attain the same error probability of 2-PAM; a factor of 3 dB.

(iii) Using the previous result, provide an upper bound to the probability of error with arbitrary M.
We have that for arbitrary M

$$p_e \leq (M-1)\mathbb{P}[n_2 - n_1 \geq \sqrt{E_\mathrm{s}}] \tag{15}$$

$$= (M-1)Q\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right). \tag{16}$$

(iv) Assume that $M = 2^k$, where k is a positive integer. Show that if $k \to \infty$, the error probability vanishes exponentially as long as

$$\frac{E_\mathrm{b}}{N_0} > 2\ln 2.$$

You may use the inequality $Q(x) \leq \frac{1}{2}e^{-\frac{x^2}{2}}$.
From the above,

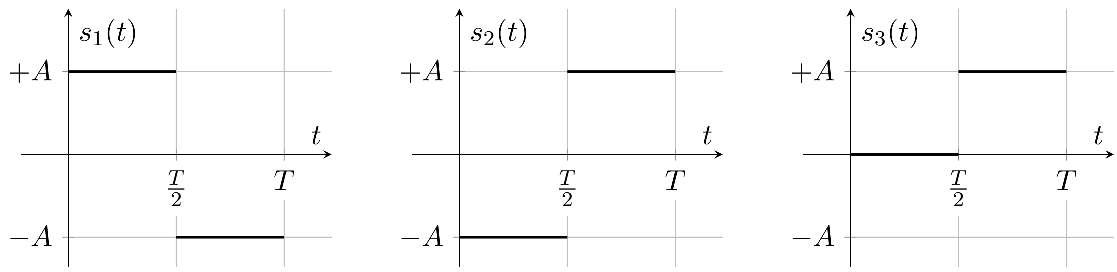
$$\begin{aligned} p_e &\leq (M-1)Q\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) && (17)\\ &\leq \frac{M-1}{2}e^{-\frac{1}{2}\frac{E_\mathrm{s}}{N_0}} && (18)\\ &< Me^{-\frac{1}{2}k\frac{E_\mathrm{b}}{N_0}} && (19)\\ &= 2^k e^{-\frac{1}{2}k\frac{E_\mathrm{b}}{N_0}} && (20)\\ &= e^{-\frac{k}{2}(\frac{E_\mathrm{b}}{N_0}-2\ln 2)}. && (21)\end{aligned}$$

Thus, as long as $\frac{E_\mathrm{b}}{N_0} > 2\ln 2$ the error probability vanishes exponentially with k.

Part (a) was done well mostly by everyone. Part (b) was done generally well, although some did not properly specify the constants of the orthonormal basis. Most chose the shifted pulse basis, and few chose the scaled signal set. Part (c).(i) was rarely answered well, most wrote what they knew about maximum likelihood detection or wrote the final result without justification, but did not derive the optimal detector. Part (c).(ii) was generally well-answered, although some candidates found incorrect shortcuts to prove the result. Part (c).(iii) was generally well-answered but several failed to explain and apply the union bound. Part (c).(iv) was generally well-answered.

Question 2

(a) Consider the signal set shown below.



(i) Determine the dimension of the signal space and find an orthonormal basis using the Gramm-Schmidt procedure. The dimension of the signal space is 2. We start the Gramm-Schmidt procedure with $s_1(t)$. Since the $\|s_1(t)\| = A\sqrt{T}$, we have that

$$f_1(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & \frac{T}{2} < t \leq T \\ 0 & \text{otherwise} \end{cases}$$

We continue with

$$\begin{aligned} g_2(t) &= s_2(t) - \langle s_2(t), f_1(t)\rangle \cdot f_1(t) && (22)\\ f_2(t) &= \frac{g_2(t)}{\|g_2(t)\|}. && (23)\end{aligned}$$

We have that

$$\langle s_2(t), f_1(t)\rangle = -\frac{1}{\sqrt{T}} \cdot A \cdot T = -A\sqrt{T}$$

Then, $g_2(t) = 0$, and thus $f_2(t) = 0$. We continue with the third signal

$$\begin{aligned} g_3(t) &= x_3(t) - \langle x_3(t), f_1(t)\rangle \cdot f_1(t) - \langle x_3(t), f_2(t)\rangle \cdot f_2(t) && (24)\\ &= x_3(t) - \langle x_3(t), f_1(t)\rangle \cdot f_1(t) && (25)\\ f_3(t) &= \frac{g_3(t)}{\|g_3(t)\|}. && (26) \end{aligned}$$

We have that

$$\langle x_3(t), f_1(t)\rangle = -\frac{A\sqrt{T}}{2}.$$

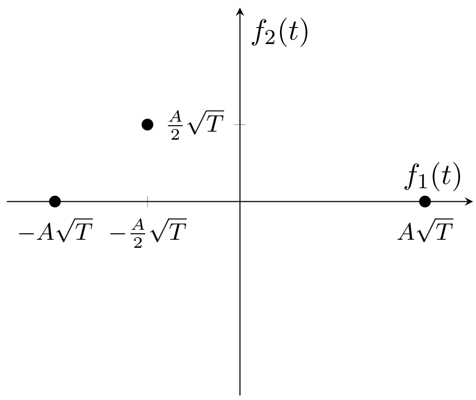
Thus,

$$g_3(t) = \begin{cases} \frac{A}{2} & 0 \le t \le \frac{T}{2} \\ \frac{A}{2} & \frac{T}{2} < t \le T \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$f_3(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le T \\ 0 & \text{otherwise.} \end{cases}$$

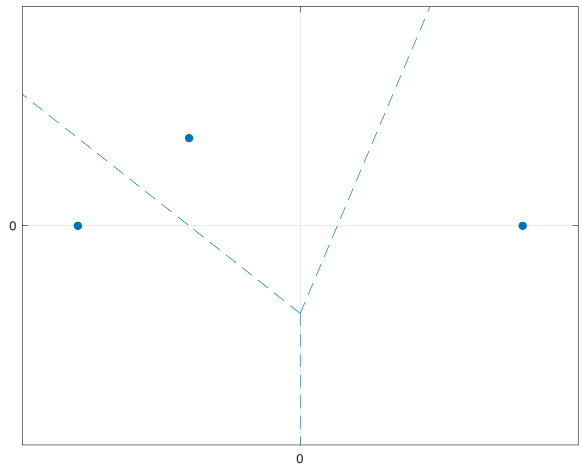
(ii) Draw the signal space and calculate the average energy per symbol E_s. The constellation is given below



The average energy is equal to

$$E_\mathrm{s} = \frac{1}{3}\left(2A^2T + \frac{1}{2}A^2T\right) = \frac{5}{6}A^2T.$$

(iii) Sketch the optimal decision regions with equiprobable symbols and derive an upper bound to the error probability. Express the result in terms of the average energy of the signal set and of the Q-function, where $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{u^2}{2}}\,du$.
The decision regions are shown in the figure below



We use the union bound for the error probability to obtain

$$p_e \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{j \neq i} Q\left(\sqrt{\frac{\|\boldsymbol{s}_i - \boldsymbol{s}_j\|^2}{2N_0}}\right).$$

We now calculate each of the distances, termed d_{ij} for simplicity.

$$d_{12} = \|\boldsymbol{s}_1 - \boldsymbol{s}_2\|^2 = 4A^2T \tag{27}$$

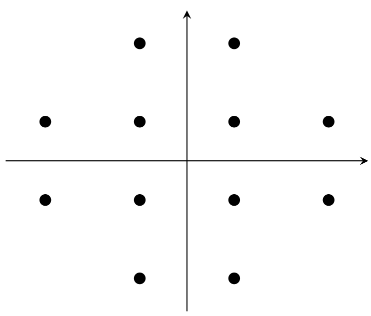
$$d_{13} = \|\boldsymbol{s}_1 - \boldsymbol{s}_3\|^2 = \frac{5}{2}A^2T \tag{28}$$

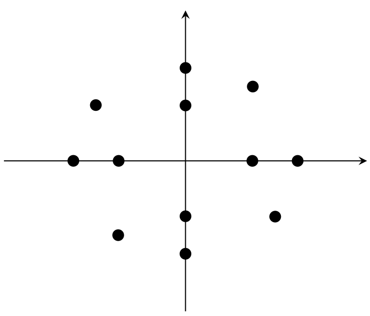
$$d_{23} = \|\boldsymbol{s}_2 - \boldsymbol{s}_3\|^2 = \frac{1}{2}A^2T \tag{29}$$

$$p_e \leq \frac{2}{3}\left(Q\left(\sqrt{\frac{4A^2T}{2N_0}}\right) + Q\left(\sqrt{\frac{\frac{5}{2}A^2T}{2N_0}}\right) + Q\left(\sqrt{\frac{\frac{1}{2}A^2T}{2N_0}}\right)\right) \tag{30}$$

$$= \frac{2}{3}\left(Q\left(\sqrt{\frac{12E_\mathrm{s}}{5N_0}}\right) + Q\left(\sqrt{\frac{3E_\mathrm{s}}{2N_0}}\right) + Q\left(\sqrt{\frac{3E_\mathrm{s}}{10N_0}}\right)\right). \tag{31}$$

(b) The 12-QAM constellation shown on the figure below (left) is used for transmission over a communications channel. In absence of noise, the received constellation is shown on the figure below (right).

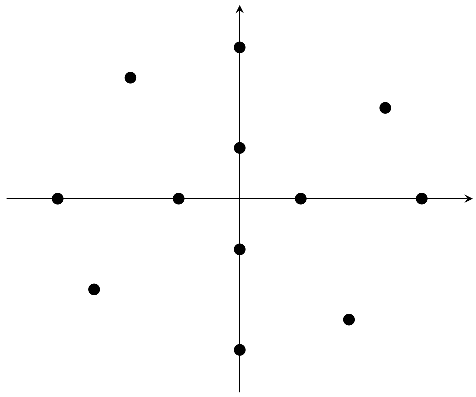




(i) Discuss the input-output characteristics of the channel.

If we observe the inner 4 points on the original constellation, they have simply been rotated $\frac{\pi}{4}$. The outer points (which all lie on a circle) have suffered not only a rotation but also an attenuation. This channel has therefore introduced different phase-shifts and attenuations to different input energies, and therefore has non-linear characteristics.

(ii) What operations would the transmitter need to do in order to compensate for the channel effects and receive the original 12-QAM constellation? Sketch the transmitted constellation. The transmitter would need to rotate the inner points by $-\frac{\pi}{4}$ as well as the outer points by the negative of the corresponding angle. In addition, the transmitter would need to multiply the magnitude of the outer points by the inverse of the attenuation introduced. The resulting transmitted constellation is given below.



Part (a).(i) was done correctly by many candidates, but several did not apply the Gramm-Schmidt procedure correctly despite knowing what specific basis they wanted to obtain. The question admitted several correct answers. Part (a).(ii) was generally well-answered, although some of mistakes from Part (a).(i) propagated to Part (a).(ii). Part (a).(iii) was turned out to be very difficult. Many candidates correctly sketched the optimal decision regions of their constellations, but found it difficult to calculate the probability of error. Part (b).(i) was generally well-answered, although several candidates only pointed to a phase rotation and did not mention the attenuation introduced by the channel. Part (b).(ii) was answered well by many of those who answered well Part (b).(i). Several candidates explained equalisation and OFDM but the answer was not connected with either.

Question 3

(a) Consider a PAM system with pulse shape $p(t)$ where the transmitted signal is $x(t) = \sum_{k=-\infty}^{\infty} X_k p(t-kT)$, where X_k are chosen from a constellation. The received signal $y(t)$ is filtered through a low-pass filter with impulse response $q(t)$.

(i) Write down the expression for the output of the low-pass filter. The output of the low-pass filter is given by

$$
\begin{aligned}
r(t) &= x(t) \star q(t) && (32)\\
&= \int_{-\infty}^{\infty} q(u)x(t-u)du && (33)\\
&= \sum_{k=-\infty}^{\infty} X_k \int_{-\infty}^{\infty} q(u)p(t-kT-u)du && (34)\\
&= \sum_{k=-\infty}^{\infty} X_k g(t-kT) && (35)
\end{aligned}
$$

where $g(t) = q(t) \star p(t)$ is the overall filter.

(ii) Write the time- and frequency-domain conditions that the pulse and low-pass filters need to fulfill in order not to have inter-symbol interference.
If we sample the output of the filter at mT, we obtain

$$r(mT) = \sum_{k=-\infty}^{\infty} X_k g((m-k)T) \tag{36}$$

there will be no inter-symbol interference if

$$g(mT) = \begin{cases} 1 & m = 0 \\ 0 & \text{otherwise} \end{cases}$$

In the frequency domain this is precisely the Nyquist criterion. The combined pulse $g(t)$ needs to be such that

$$\sum_{n=-\infty}^{\infty} G(f - \frac{n}{T}) = T.$$

(iii) Consider a frequency-selective channel with impulse response

$$h(t) = \sum_{\ell=1}^{L} \alpha_\ell \delta(t - \tau_\ell).$$

Explain the main impairment this channel introduces. Briefly explain the main methods to deal with this channel impairment.

This impulse response will most likely cause inter-symbol interference (ISI), especially if $T < \tau_L$. There are essentially 2 methods to combat ISI:

i. Equalisation: this is done by processing the received signal at the receiver trying to remove or mitigate the effect of ISI. There are two types of equalisers, zero-forcing and MMSE. The first removes ISI but might enhance the noise; MMSE finds a better tradeoff between reducing ISI and noise.

ii. OFDM: this requires a complete re-design of the end-to-end communications system. OFDM operates by converting the ISI channel into a set of parallel channels, each with a flat response.

(b) Describe briefly the difference between distance-vector protocols and link-state protocols for routing. Give an example for each and state which routing algorithm it uses.

Distance vector protocols: broadcast distances to destinations that you know to your neighbours, eg, RIP uses Bellman-Ford.

Link-state: send distances to your neighbours to everyone, e.g., OSPF uses Dijkstra.

(c) Consider a network with 3 nodes (A, B and C). Shortest path routing is used where the weight of each link depends on it's capacity and the flow on that link. The initial weight of the links from from A to B and C to A, and from B to C and C to B, are 1. There is no direct link beween A and C. Due to a change in flow, the weight of the link from B to C suddenly changes to 100. Describe how the routing information is subsequently updated in the case of:

(a) A link state protocol.

B sends updated distance to C to nodes A and C, who recompute routing tables correctly.

(b) A distance vector protocol (assume in this case that B receives a packet from A containing its distance vector at the same instance at the weight changes). A advertises to B a route of length 2 to C, this is better than the new distance 100, so B sends packets addressed to C to node A, and advertises a route of length 3 to C (via A). A now advertises a route of length 4, etc. This continues on until A advertises a route of length 100, and until this is time all packets aimed at C from A or B just get passed back and forth between A and B.

(d) What undesirable interaction been congestion control and routing can occur even in situations where a link state protocol is used ?

Route flapping: congested routes get a longer length and so working algorithms move flows away from them. Then, the new routes get congested and flow switches back.

Generally answered well by those that took it. Part (a).(i) was generally well-answered, although several candidates did not write the expression correctly despite being in the notes. Parts (a).(ii) and (a)(iii) were generally well-answered. Part (b) was generally well-answered, as was Part (c).(i). Few candidates answered Part (c).(iii) correctly and a common mistake was to arrive at an update after 2 iterations.

Question 4

Consider the convolutional encoder in Fig. 1.

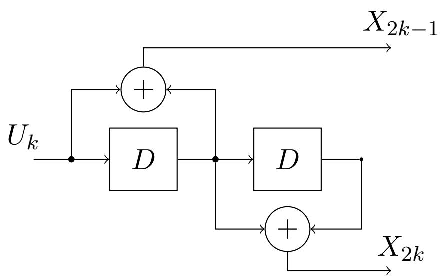
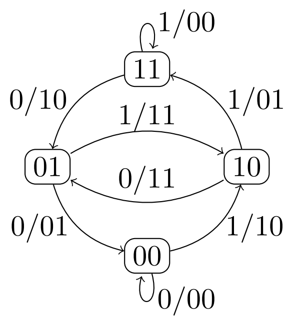


Figure 1: Figure 1

(a) What is the rate of the code?

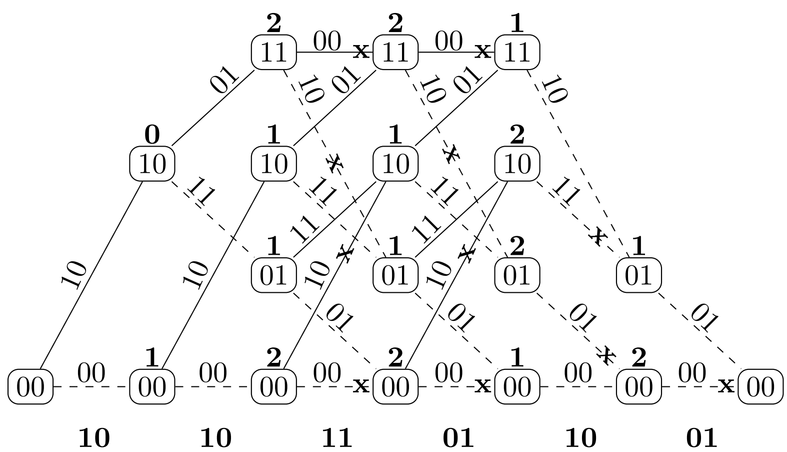
The code has rate $R = 1/2$ because it emits two code digits X_{2k-1}, X_{2k} for every input digit U_k.

(b) Draw the state diagram of the encoder, labelling each branch with the input and output bits corresponding to that branch.



(c) A sequence of four symbols followed by two termination zeros is transmitted over a binary symmetric channel (BSC) and the received sequence is 10, 10, 11, 01, 10, 01. Determine a maximum likelihood code sequence and the corresponding information sequence for this received sequence.

We use the Viterbi algorithm to decode



and conclude that the unique maximum likelihood sequence is $10, 11, 11, 01, 10, 01$ and the corresponding information sequence is $1, 0, 1, 1$.

(d) The bit and block error frequencies for a Viterbi decoder and a Bahl Cocke Jelinek Raviv (BCJR) decoder were measured through simulation for long data blocks and recorded as $f_1 = 0.000126$, $f_2 = 0.000174$, $f_3 = 0.0346$, $f_4 = 0.0375$, but we forgot to label the measurements. Determine which of the four frequencies f_1, f_2, f_3 and f_4 corresponds to which measurement, i.e., (Viterbi, block error), (Viterbi, bit error), (BCJR, block error) and (BCJR, bit error) and justify your answer.

The two smaller values are clearly the bit error frequencies and the two larger the block error frequencies, because any single bit error in a block would lead to a block error. The BCJR is optimal in terms of bit error probability and the Viterbi algorithm is optimal in terms of block error probability, resulting in the assigments

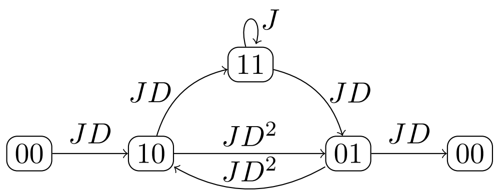
$$\begin{cases} f_1 \longrightarrow \text{(bit error, BCJR)} \\ f_2 \longrightarrow \text{(bit error, Viterbi)} \\ f_3 \longrightarrow \text{(block error, Viterbi)} \\ f_4 \longrightarrow \text{(block error, BCJR)} \end{cases}$$

(e) Is the encoder catastrophic? Justify your answer.

The encoder is catastrophic because its connection polynomials are $(1+D)$ and $D+D^2 = D(1+D)$ and the greatest common divisor of these two polynomials is $1 + D$ which cannot be expressed in the form D^ℓ and hence corresponds to a catastrophic encoder according to the Massey-Sain theorem.

(f) Determine the free distance d_{free} of the code and specify what length input sequences generate code sequences of weight d_{free}? Explain if and how your findings are consistent with your answer to part (e).

We split the zero state to obtain a transfer diagram from/to the zero state, labeling the branches with JD^w where w is the Hamming weight of the corresponding output,



and obtain the transfer function by solving a system of equations or using Mason's gain formula for signal flow graphs

$$\begin{aligned} T(D, J) &= \frac{J^3 D^4}{1 - J - J^2 D^4} \\ &= J^3 D^4 (1 + (J + J^2 D^4) + (J + J^2 D^4)^2 + (J + J^2 D^4)^3 + \ldots) \\ &= D^4 (J^3 + J^4 + J^5 + \ldots) + o(D^8) \end{aligned}$$

showing that $d_{\text{free}} = 4$ and there are paths of lengths $3, 4, 5, 6, \ldots$ that generate codes sequences of weight 4. This is consistent with our statement that the encoder is catastophic, since there are input sequences of any length that map to a codeword of weight 4, and hence with only 2 channel errors we can cause an unbounded number of errors in the input sequence estimate.

Most students got the rate right and almost everyone was able to draw an accurate state diagram of the convolutional encoder. Performance on the Viterbi algorithm in part (c) was variable, with *many having not completely understood the meaning of a terminated code (the last two 0s terminate the encoder back to the all-zero state. They are NOT information symbols). Many struggled to rank the error probabilities of decoders in part (d). Students were expected to apply common sense to understand that block error rates are an order of magnitude higher than bit error rates. Not many did, but any answer that had a lower block error rate for the block-error optimal Viterbi algorithm and a lower bit error rate for the bit-error optimal BCJR was accepted as correct. The last two parts were more advanced. There were quite a few students who did them perfectly but also many who did not attempt these parts.*