3F4 Data Transmission 2023
Crib

Question 1

(a) The encoder has rate $R = 1/3$ because it emits 3 code digits for every input digit. Its description in octal notation is $(11, 12, 17)_8$.

(b) The encoder has $2^3 = 8$ states. The transitions from the state $(1, 1, 0)$ lead to:

$$\begin{cases} \text{state } (0,1,1) \text{ for input } 0 \text{ with corresponding outputs } (0,1,0) \\ \text{state } (1,1,1) \text{ for input } 1 \text{ with corresponding outputs } (1,0,1) \end{cases}$$

(c) See attached Viterbi algorithm with computed Hamming distance metrics. The recovered maximum likelihood (ML) code sequence is 111,110,101,000,000,111,110,101 corresponding to the information sequence 1,1,1,1,1. The solution is unique as there are no ties on the winning path.

(d) (a)

$$\begin{aligned} \mathrm{P}(U = 1) &= \frac{\sum_{i,j \text{ such that } u=1} \alpha_i \gamma_{ij} \beta_j}{\sum_{i,j} \alpha_i \gamma_{ij} \beta_j} \\ &= \frac{3+2+5+2+4+2+1+3}{1+3+4+2+1+5+2+2+3+4+6+2+3+1+5+3} \\ &= \frac{22}{47} = 0.468 \end{aligned}$$

(b)

$$\begin{aligned} \mathrm{P}(X_2 = 1) &= \frac{\sum_{i,j \text{ such that } x_2=1} \alpha_i \gamma_{ij} \beta_j}{\sum_{i,j} \alpha_i \gamma_{ij} \beta_j} \\ &= \frac{3+2+1+2+4+6+3+5}{47} = \frac{26}{47} = 0.553 \end{aligned}$$

(e) The encoder is catastrophic because its connection polynomials are $1 + D^3, 1 + D^2$ and $1 + D + D^2 + D^3$ which are all divisible by $1 + D$ in binary. According to the Massey & Sain theorem, the greatest common divisor of the connection polynomials must be of the form D^ℓ for an encoder not to be catastrophic. For this encoder, an input sequence of all ones corresponding to a D transform of $\frac{1}{1+D} = 1 + D + D^2 + D^3 + \ldots$ would result in a finite Hamming weight sequence on all 3 outputs and hence an overall finite Hamming weight path, so that a finite number of errors may result in an infinite number of decoding errors.

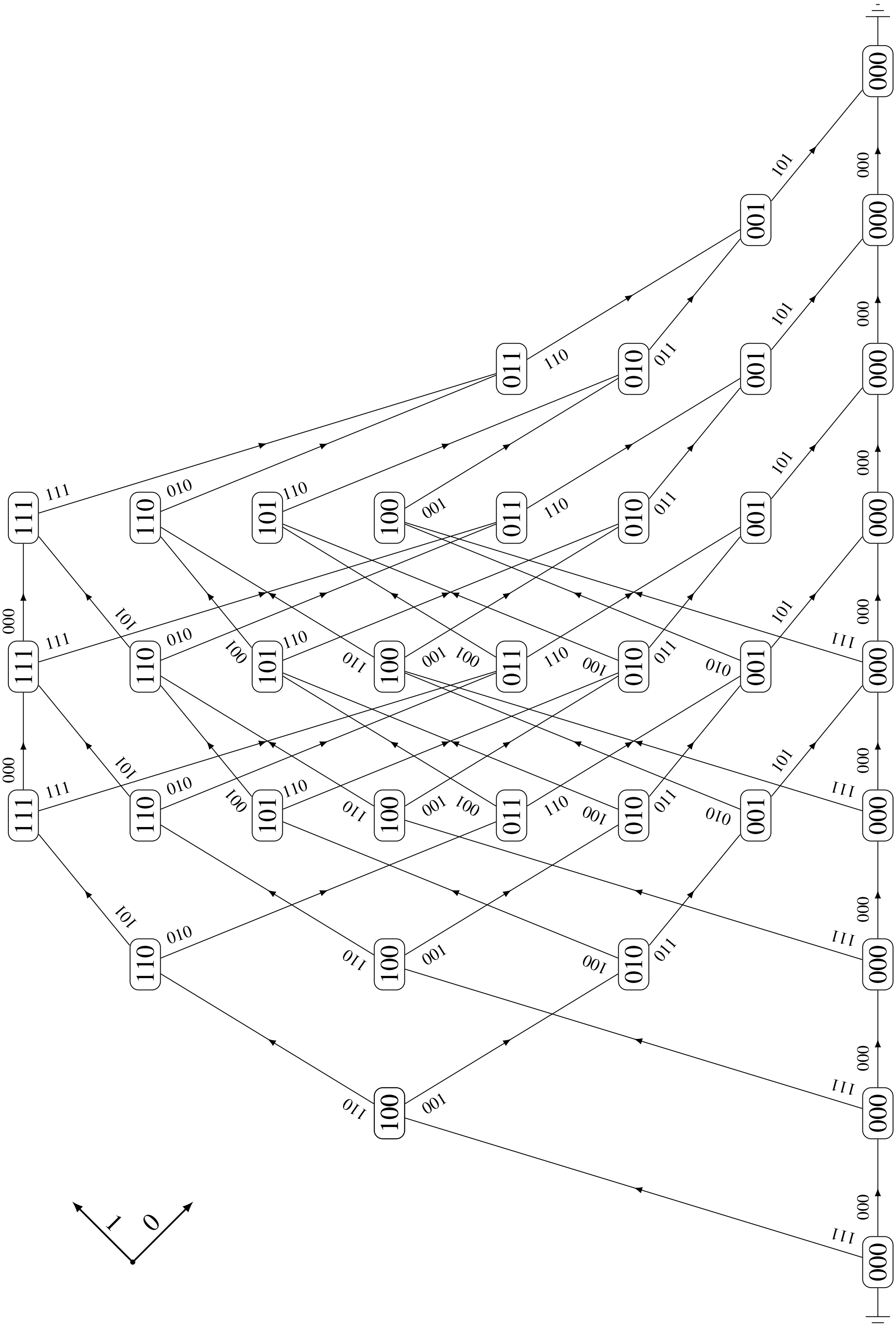


Figure 1: Trellis.

Question 2

(a) The Gram-Schmidt method constructs an orthonormal basis as follows

$$f_1(t) = \frac{s_1(t)}{\|s_1(t)\|} = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq T \\ 0 & \text{otherwise.} \end{cases}$$

The next signal is

$$f_2(t) = \frac{g_2(t)}{\|g_2(t)\|} \tag{1}$$

$$g_2(t) = s_2(t) - \langle s_2(t), f_1(t) \rangle f_1(t) \tag{2}$$

$$= s_2(t) \tag{3}$$

and thus

$$f_2(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & \frac{T}{2} < t \leq T \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$f_3(t) = \frac{g_3(t)}{\|g_3(t)\|} \tag{4}$$

$$g_3(t) = s_3(t) - \langle s_3(t), f_1(t) \rangle f_1(t) - \langle s_3(t), f_2(t) \rangle f_2(t) \tag{5}$$

$$= s_3(t) \tag{6}$$

since they are orthogonal. Thus

$$f_3(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq \frac{T}{4} \\ -\frac{1}{\sqrt{T}} & \frac{T}{4} < t \leq \frac{T}{2} \\ \frac{1}{\sqrt{T}} & \frac{T}{2} < t \leq \frac{3T}{4} \\ -\frac{1}{\sqrt{T}} & \frac{3T}{4} < t \leq T \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the dimension is $K = 3$. The signal set is orthogonal.

(b) The transmission rate is $R_b = \frac{1}{T} \log_2 3$bits/s.

(c) The vector representation is

$$\boldsymbol{s}_1 = (A\sqrt{T}, 0, 0), \boldsymbol{s}_2 = (0, A\sqrt{T}, 0), \boldsymbol{s}_3 = (0, 0, A\sqrt{T}) \tag{7}$$

The average energy is $E_s = A^2T$ and the minimum distance is $2A^2T$ since all constellation points are at the same distance.

(d) Since the signal set is orthogonal, the optimal detector is formed by a bank of correlators (or matched filter), followed by the rule

$$\hat{m} = \arg\max_{i=1,2,3} y_i$$

where y_i is the output of the ith correlator.

(e) All are equally vulnerable, since the signal set is orthogonal and all non-zero coordinates are the same.

(f) The union bound states that

$$p_e \leq \frac{1}{M} \sum_{m=1}^{M} \sum_{m' \neq m} Q\left(\sqrt{\frac{\|\boldsymbol{s}_m - \boldsymbol{s}_{m'}\|^2}{2N_0}}\right).$$

Since the signal set is orthogonal, and all signals are at the same distance, the above bound becomes

$$p_e \leq (M-1)Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right) = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

(g) The bound is valid at high SNR, as this is when the individual pairwise error events dominate.

Question 3

(a) (i) The dimension of this signal space is $K = 2$.

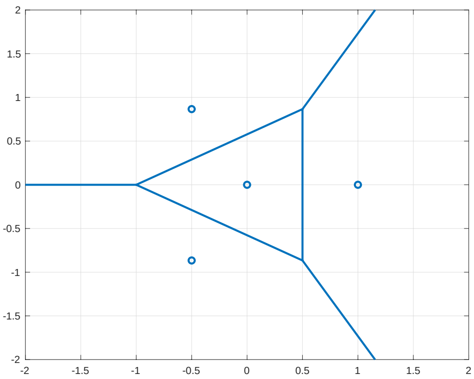
(ii) An orthonormal basis could be

$$f_1(t) = \begin{cases} \sqrt{\frac{2}{T}} & 0 \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise,} \end{cases} \qquad f_2(t) = \begin{cases} \sqrt{\frac{2}{T}} & \frac{T}{2} \leq t \leq T \\ 0 & \text{otherwise.} \end{cases}$$

A suitable signal set could be

$$s_1(t) = \begin{cases} -\sqrt{\frac{1}{2T}} & 0 \leq t \leq \frac{T}{2} \\ -\sqrt{\frac{3}{2T}} & \frac{T}{2} \leq t \leq T \\ 0 & \text{otherwise,} \end{cases} \quad s_2(t) = \begin{cases} -\sqrt{\frac{1}{2T}} & 0 \leq t \leq \frac{T}{2} \\ \sqrt{\frac{3}{2T}} & \frac{T}{2} \leq t \leq T \\ 0 & \text{otherwise,} \end{cases} \quad s_3(t) = 0 \quad s_4(t) = \begin{cases} \sqrt{\frac{2}{T}} & 0 \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

(iii) The decision regions are shown in Figure 2 (left).



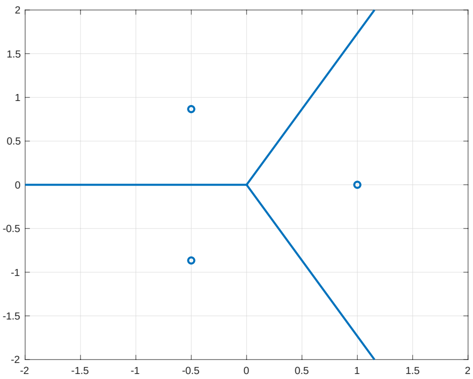


Figure 2: Decision regions for questions (a) (iii) (left) and (a) (iv) (right).

(iv) The decision regions are shown in Figure 2 (right).

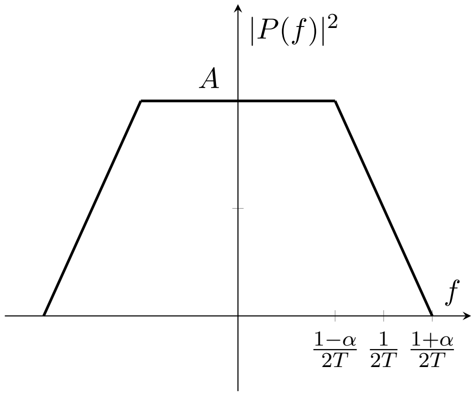
(b) (i) The constellation on the left hand side has higher minimum distance, hence, it will have a lower probability of error at high SNR.

(ii) Consider an OFDM system with N points over a channel with L inter-symbol interference taps. The cyclic prefix is a sequence of L time-domain symbols that is added at the beginning of each IDFT frame in order to make the linear convolution of the channel appear as a circular convolutions. These L symbols coincide with the last L symbols of the IDFT frame. The cyclic prefix lowers the data rate by a factor $\frac{N}{N+L}$.

(iii) The constellation on the right has itself the best possible peak-to-average power ratio, as all symbols have maximum energy. Therefore, this will have the lowest peak-to-average power ratio at the output of the IDFT.

Question 4

(a) (i) The magnitude squared of the Fourier transform of the pulse $|P(f)|$ is sketched below.



(ii) By Parseval we have that the energy of the pulse E_p is

$$E_p = \int |p(t)|^2 dt = \int |P(f)|^2 df.$$

Therefore, we have that

$$E_p = \frac{A}{T}(1-\alpha) + \frac{A}{T}\alpha = \frac{A}{T}$$

Thus, in order for $E_p = 1$ we need $A = T$.

(iii) The Nyquist criteria for no-ISI says that the overall pulse $g(t)$ needs to be such that

$$\sum_n G\left(f - \frac{n}{T}\right) = T.$$

For $\alpha = 0$ we have that $G(f) = P(f)^2$ is a rectangular pulse of amplitude T and duration $\frac{1}{T}$ centered at the origin. Thus, the Nyquist criterion is satisfied as the shifted pulses result in a constant T.

Similarly, for $\alpha = 1$ we have that $G(f) = P(f)^2$ is a triangular pulse centered at the origin, duration $\frac{2}{T}$ and maximum amplitude T. Shifting these pulses each multiple of $\frac{1}{T}$ also yields a constant at T, and therefore satisfies the Nyquist criterion.

There is no ISI for either value of α.

(iv) For $\alpha = 0$, we have that $g(t) = \text{sinc}(\frac{\pi t}{T})$. For $\alpha = 1$, $G(f)$ is a triangular pulse, which means that it is the convolution of two rectangular pulses in frequency. Thus, $g(t) = \text{sinc}^2(\frac{\pi t}{T})$.

(b) See next page

a) i) $w \Rightarrow w + \frac{1}{w}$

ii) $w \Rightarrow \frac{w}{2}$

b) $E(w(t')) = P_{loss} \cdot (w(t') \| loss) + P_{in\ sequence} \, (w(t') |_{in\ sequence})$

$= p \frac{w}{2} + (1-p)\left(w + \frac{1}{w}\right)$ as required

In equilibrium, $E(w(t')) = w(t)$

ie $w = \frac{pw}{2} + (1-p)\left(w + \frac{1}{w}\right)$

$w = \frac{pw}{2} + w - pw + \frac{1}{w} - \frac{p}{w}$

ie $\frac{pw}{2} = \frac{1-p}{w} \iff w^2 = \frac{2(1-p)}{p}$

$\Rightarrow w = \sqrt{\frac{2(1-p)}{p}}$

$w = 500 \cdot 10^6 \cdot 0.1 = 50 \cdot 10^6$

$pw^2 = 2 - 2p \Rightarrow p(2+w^2) = 2 \Rightarrow p = \frac{2}{2+w^2} \approx \frac{2}{w^2}$

$= 8 \cdot 10^{-16}$

$\Rightarrow 500 \cdot 10^6 \times 8 \cdot 10^{-16} = 4 \cdot 10^{-7}$ loss events / second allowed

$\approx 1 / 700$ hours !

(ie 1 loss per 700 hours on average)

This is not reasonable. Even without congestion hardware errors cause the occasional drop c)

$$W = p\frac{W}{2} + (1-p)\left(W + \frac{1}{20}\right)$$

$$= \frac{pW}{2} + W - \frac{pW}{2} + \frac{1-p}{20}$$

$$\Rightarrow \frac{pW}{2} = \frac{1-p}{20} \qquad \Rightarrow \qquad W = \frac{1-p}{10p}$$

$$p\left(\frac{W}{2} + \frac{1}{20}\right) = \frac{1}{20}$$

$$p \approx \frac{1}{10W} \approx \frac{1}{10 \cdot 50 \cdot 10^{6}} = 10^{-8}$$

$$500 \cdot 10^{6} \cdot 10^{-8} = 5 \text{ losses per second.}$$

This is reasonable, a greater number than would be expected from hardware errors and so would be being caused by congestion.