Version AGiF/2

EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 30 April 2025 9.30 to 11.10

Module 3F4

DATA TRANSMISSION

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed.
Engineering Data Book.

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) Draw the encoder for the $(6,7)_8$ convolutional code. [10%]

(b) Draw a state diagram for the encoder. [10%]

(c) A 3 bit sequence $\mathbf{U} = [U_1, U_2, U_3]$ has been encoded with no termination bits, resulting in a code sequence $\mathbf{X} = [X_1, X_2, X_3, X_4, X_5, X_6]$. This has been transmitted over a Binary Symmetric Channel (BSC) with transition probability $p = 0.1$ and received as $\mathbf{y} = [y_1, y_2, y_3, y_4, y_5, y_6] = [0, 1, 1, 0, 1, 0]$. What is $P_{\mathbf{Y}|\mathbf{U}}(\mathbf{y}|[0,0,0])$? [5%]

(d) For $\mathbf{y}$ as given in part (c), compute $\max_{\mathbf{u}} P_{\mathbf{Y}|\mathbf{U}}(\mathbf{y}|\mathbf{u})$ and specify the full set of vectors $\mathbf{u}$ that maximises this expression. [15%]

(e) For $\mathbf{y}$ as given in part (c), compute $P_{U_3|\mathbf{Y}}(1|\mathbf{y})$ and $P_{X_5|\mathbf{Y}}(1|\mathbf{y})$ using the BCJR (forward-backward) algorithm. [30%]

(f) Express the detour enumerator transfer function $T(J, D, N)$ for the encoder and hence determine the free distance d_{free} of the encoder. [30%]

2 (a) Consider 2-PAM transmission with independent and equiprobable symbols $a[k] \in \{-A, A\}$ over the discrete-time channel whose output

$$y[k] = x[k] + 2x[k-1] + \frac{1}{2}x[k-2] + z[k]$$

where $z[k]$ are AWGN samples with zero mean and variance $\frac{N_0}{2}$.

(i) In the absence of noise, find the channel impulse response $h[k]$. [10%]

(ii) In the absence of noise, find the received signal constellation. [20%]

(iii) In the absence of noise, what type of filter at the receiver would result in the best performance? Justify your answer. [10%]

(iv) In the presence of noise, what type of filter at the receiver would result in the best performance? Justify your answer. [10%]

(b) Consider transmission of a binary linear code of length n and dimension $\frac{n}{2}$ with BPSK modulation over an AWGN channel. Assume that the Hamming weight spectrum of the code is given by A_w, the number of codewords at Hamming weight w. Let $d_{\min}$ denote the minimum distance of the code.

(i) Show that the word error probability of the code can be bounded as

$$p_e \leq \sum_w A_w Q\left(\sqrt{\frac{2wE_b}{N_0}}\right)$$

[25%]

(ii) Show that with fast Rayleigh fading, at high SNR the error probability behaves as

$$p_e \leq A_{d_{\min}} \left(\frac{1}{1 + \frac{E_b}{N_0}}\right)^{d_{\min}}$$

[15%]

(iii) If the code is now mapped on to a 16-QAM modulation, what is the rate of the overall transmission scheme? [10%]

3 Consider the three waveforms $f_n(t)$ shown in Fig. 1 below,

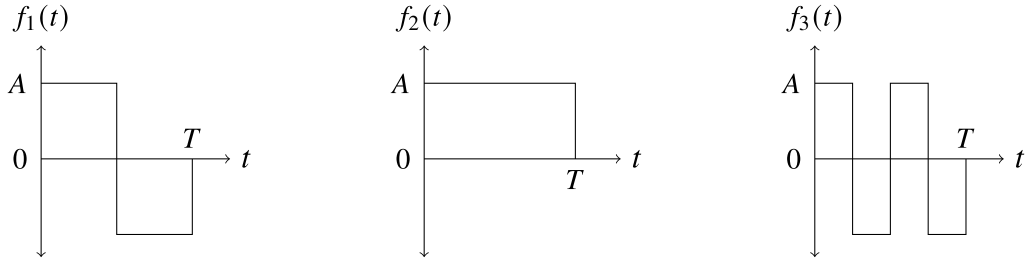

Fig. 1

(a) What is the dimension of the signal set in Fig. 1? [5%]

(b) Are the waveforms in Fig. 1 orthonormal? Justify your answer. [10%]

(c) Calculate the average energy of the signal set described by the waveforms $f_n(t)$, for $n = 1, 2, 3$. [10%]

(d) Draw the signal constellation using an orthonormal basis and explain how to find the decision regions for optimal detection. [10%]

(e) Is it possible to express the waveform $x(t)$ below as a linear combination of the waveforms $f_n(t)$? Justify your answer. [15%]

$$x(t) = \begin{cases} -1 & 0 \leq t < \frac{T}{4} \\ +1 & \frac{T}{4} \leq t < \frac{3T}{4} \\ -1 & \frac{3T}{4} \leq t \leq 1 \end{cases}$$

(f) Consider the following signal set

$$s_1(t) = \sqrt{E_s}\sqrt{\frac{2}{T}}\cos\left(2\pi\left(f_c - \Delta_f\right)t\right)$$
$$s_2(t) = \sqrt{E_s}\sqrt{\frac{2}{T}}\cos\left(2\pi f_c t\right)$$
$$s_3(t) = \sqrt{E_s}\sqrt{\frac{2}{T}}\cos\left(2\pi\left(f_c + \Delta_f\right)t\right)$$

with $f_c T = \frac{K}{2}$, $\Delta_f T = \frac{1}{2}$ and K a suitably large integer. The signal set is used for transmission over an AWGN channel where the noise process $n(t)$ is assumed to have zero mean, i.e. $\mathbb{E}[n(t)] = 0$, and autocorrelation function

$$\mathbb{E}[n(t)n(t+\tau)^*] = \frac{N_0}{2}\delta(\tau), \qquad \text{for all } t, \tau.$$

(i) What is the dimension of the signal space? [10%]

(ii) Provide the vector representation of the signal space and draw the signal constellation. [10%]

(iii) Derive the optimum demodulator and draw the corresponding block diagram. Justify your answer. [15%]

(iv) Show that the error probability of the optimum demodulator can be bounded as

$$p_e \leq 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

Justify the steps of your derivation. [15%]

4 Consider a system with 2 transmit and 1 receive antennas. The baseband received signal for each symbol instant mT, where T is the symbol period, can be formulated as

$$y(m) = h_1(m)x_1(m) + h_2(m)x_2(m) + w(m) \tag{1}$$

where the noise samples $w(m) \sim \mathcal{N}(0, N_0)$ are i.i.d. complex Gaussian random variables with zero mean and variance N_0. For $i = 1, 2$, $h_i(m)$ are the flat fading coefficients from transmit antenna i, with $h_1(m)$ and $h_2(m)$ independent, and $x_i(m)$ are the transmitted symbols from antenna i. The Alamouti space-time code is employed for which $x_1(m) = u_1(m)$, $x_2(m) = u_2(m)$, $x_1(m+1) = -u_2^*(m)$, $x_2(m+1) = u_1^*(m)$, where the asterisk denotes complex conjugation.

(a) Assuming the channel remains constant for more than two symbol intervals, show that the received symbols for two consecutive symbol periods can be expressed as [10%]

$$\boldsymbol{y} = \begin{bmatrix} y(m) \\ y^*(m+1) \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \cdot \begin{bmatrix} u_1(m) \\ u_2(m) \end{bmatrix} + \begin{bmatrix} w(m) \\ w^*(m+1) \end{bmatrix} = \boldsymbol{Hx} + \boldsymbol{w}$$

(b) Show that $\boldsymbol{H}^H\boldsymbol{H} = \gamma\boldsymbol{I}$ where $\gamma = |h_1|^2 + |h_2|^2$ where the superscript H denotes Hermitian. [10%]

(c) Show that the filtered noise vector $\boldsymbol{H}^H\boldsymbol{w}$ is such that $\mathbb{E}[\boldsymbol{H}^H\boldsymbol{w}] = \boldsymbol{0}$ and $\mathbb{E}[\boldsymbol{H}^H\boldsymbol{w}\boldsymbol{w}^H\boldsymbol{H}] = \gamma N_0\boldsymbol{I}$. [10%]

(d) If the best possible detector is employed, show that the decision criterion is equivalent to [30%]

$$\arg\min_{u_i} |r_i - \sqrt{\gamma}u_i|^2, \; i = 1, 2 \quad \text{with} \quad \boldsymbol{r} = \frac{1}{\sqrt{\gamma}}\boldsymbol{H}^H\boldsymbol{y}$$

(e) Consider the channel in (1) assuming $h_i = e^{j\theta_i}, i = 1, 2$, with θ_i known at the receiver. Calculate the probability of error of the scheme with QPSK modulation. Is there any gain with respect to the case of a single antenna? [20%]

(f) Consider now that the channels between the transmit and receive antennas are independent and Rayleigh fading. Provide an upper bound to the probability of error. What is the diversity gain? [15%]

(g) Propose a simple solution to increase the diversity gain. [5%]

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