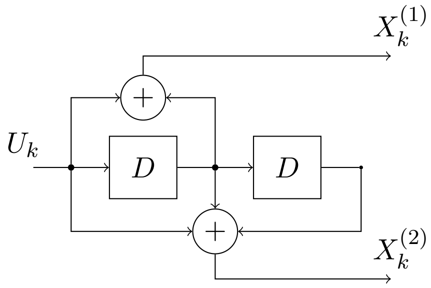
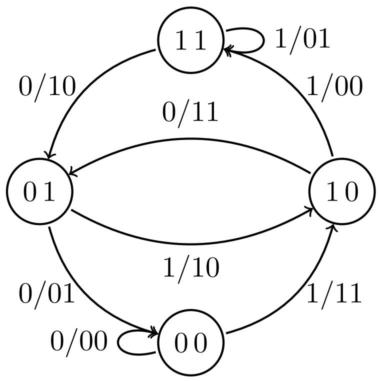
3F4 Data Transmission 2025
Crib

1. (a) We draw the encoder:

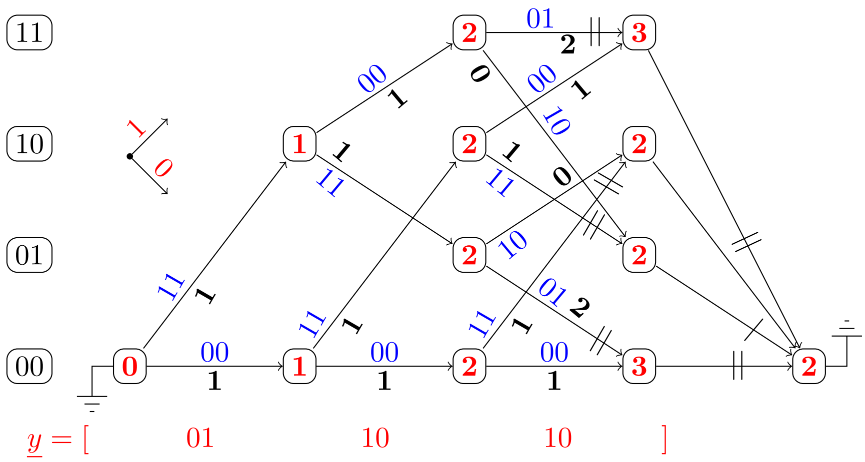


(b) We draw the state diagram of the encoder:

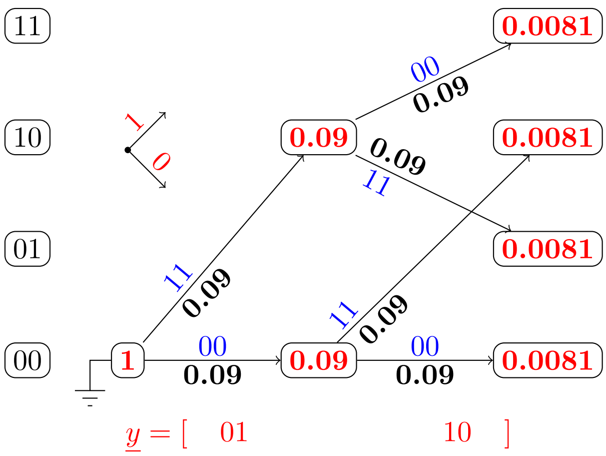


(c) $P_{\boldsymbol{Y}|\boldsymbol{U}}(\boldsymbol{y}|[0,0,0]) = P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|[0,0,0,0,0,0]) = p^3(1-p)^3 = 0.000729$

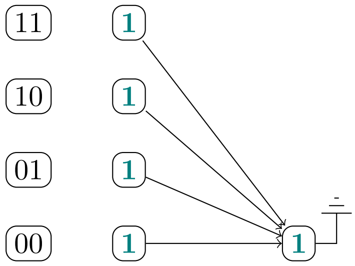
(d) We run the Viterbi algorithm, which has a tie so there are two maximum likelihood paths: $\boldsymbol{u} = [1,0,1]$ and $\boldsymbol{u} = [1,1,0]$, for wich $P_{\boldsymbol{Y}|\boldsymbol{U}}(\boldsymbol{y}|\boldsymbol{u}) = p^2(1-p)^4 = 0.006561$.



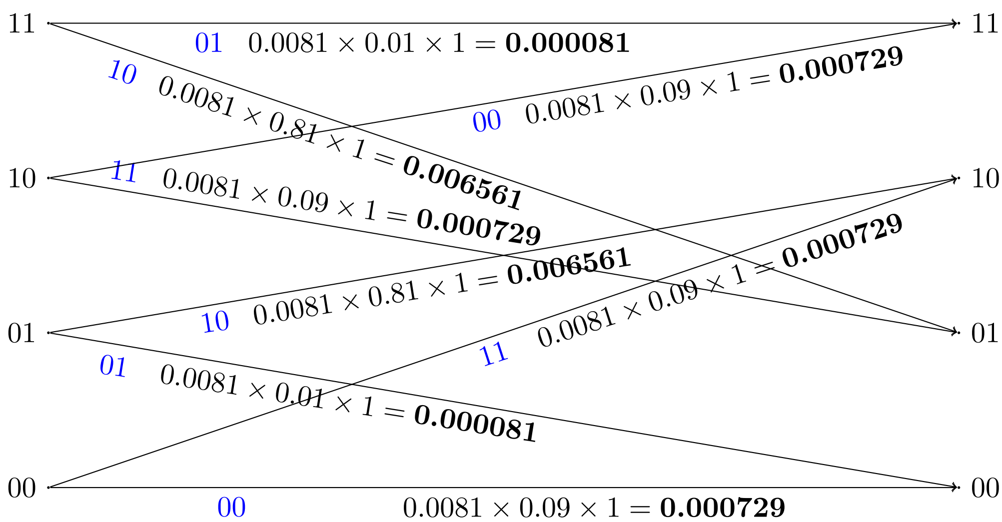
(e) We first compute the α's up to stage 2:



then compute the β's up to stage 3:

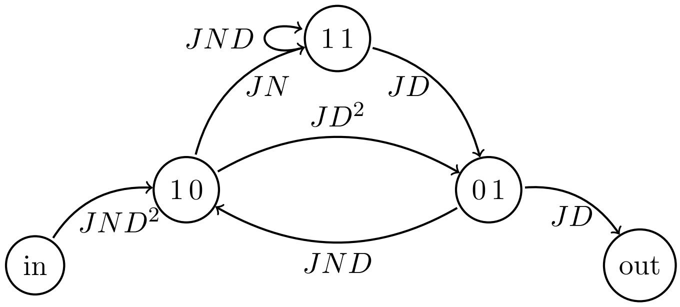


For the summary stage, we only need to compute the $\alpha_i \gamma_{ij} \beta_j$ for the relevant trellis stage that contains U_3 and X_5:



$P_{U_3|\boldsymbol{Y}}(1|\boldsymbol{y}) = \frac{0.006561+2\times0.000729+0.000081}{0.0162} = \frac{1}{2}$ and $P_{X_5|\boldsymbol{Y}}(1|\boldsymbol{y}) = \frac{2\times0.006561+2\times0.000729}{0.0162} = 0.9.$

(f) We re-draw the state diagram, splitting the zero state into an “in” and an “out” state and label with JN^nD^d where n is the Hamming weight of the information symbol and d is the Hamming weight of the code sequence for each path.



We can now obtain the transfer function either by solving a sytem of equations or using Mason's rule

$$T(J,N,D) = \frac{J^3ND^5(1-JND)+J^4N^2D^4}{1-J^2ND^3-JND-J^3N^2D^2+J^3N^2D^4}$$

Since we're only interested in code distance, we can consider

$$T(1,1,D) = \frac{D^4+o(D^5)}{1-D-D^2-D^3+D^4} = (D^4+o(D^5))(1+o(D)) = D^4+o(D^5)$$

so the free distance of this encoder is 4 and there is only one path at distance 4 from the all zero path.

2. (a) (i) The impulse response of this ISI channel is

$$h[k] = \delta[k] + 2\delta[k-1] + \frac{1}{2}\delta[k-2]$$

(ii) In the absence of noise, the signal constellation is

$x[k]$	$x[k-1]$	$x[k-2]$	Constellation point
+A	+A	+A	3.5A
+A	+A	-A	2.5A
+A	-A	+A	-0.5A
+A	-A	-A	-1.5A
-A	+A	+A	1.5A
-A	+A	-A	0.5A
-A	-A	+A	-2.5A
-A	-A	-A	-3.5A

This is an 8-point constellation on the real line with points $\{\pm 3.5A, \pm 2.5A, \pm 1.5A, \pm 0.5A\}$.

(iii) In the absence of noise we will employ a zero forcing equaliser that will eliminate the effects of ISI.

(iv) In the presence of noise, we will resort to an MMSE filter that accounts for the noise, since the zero forcing equaliser enhances the noise.

(b) (i) Since the code is linear, we can assume the transmission of the all zero codeword, which mapped onto BPSK gives $\boldsymbol{c}_0 = (1,\dots,1)$. This is a signal space defined by all codewords. By the union bound we know that the error probability is bounded by

$$p_e \le \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i} Q\left(\sqrt{\frac{\|\boldsymbol{s}_i-\boldsymbol{s}_j\|^2}{2N_0}}\right) \tag{1}$$

$$= \sum_{\boldsymbol{c}\neq\boldsymbol{c}_0} Q\left(\sqrt{\frac{\|\boldsymbol{c}-\boldsymbol{c}_0\|^2}{2N_0}}\right) \tag{2}$$

where $\|\boldsymbol{c}-\boldsymbol{c}_0\|^2$ is a sum of w non-zero terms, each adding $4E_b$. Thus, we have that

$$p_e \leq \sum_{\boldsymbol{c}\neq\boldsymbol{c}_0} Q\left(\sqrt{\frac{2E_b w}{N_0}}\right) = \sum_w A_w Q\left(\sqrt{\frac{2E_b w}{N_0}}\right)$$

where the sum over all different codewords simplifies as a sum over all Hamming weights, as the pairwise error probability depends only on the Hamming weight of the pairwise error event.

(ii) The pairwise error probability for a given known fading sequence is

$$P(\boldsymbol{c}_0 \to \boldsymbol{c}|\boldsymbol{h}) = Q\left(\sqrt{2E_b \sum_{i=1}^{w} |h_i|^2 N_0}\right) \leq e^{-\frac{E_b}{N_0}\sum_{i=1}^{w}\alpha_i} \tag{3}$$

with $\alpha_i = |h_i|^2$ Averaging over the fading

$$\mathbb{E}_{\boldsymbol{h}}[P(\boldsymbol{c}_0 \to \boldsymbol{c}|\boldsymbol{h})] \leq \int e^{-\frac{E_b}{N_0}\sum_{i=1}^{w}\alpha_i} e^{-\sum_{i=1}^{w}\alpha_i} d\alpha 1 \dots d\alpha_w = \left(\frac{1}{1+\frac{E_b}{N_0}}\right)^w \tag{4}$$

(iii) The rate is $R = \frac{1}{2} \times 4 = 2$ bits/channel use.

3. (a) By inspection the dimension is $K = 3$.

(b) The signals are orthogonal, but not orthonormal as their energies are not one.

(c) The energy of each signal is the same

$$E_s = A^2 T$$

(d) An orthonormal basis is

$$\phi_n(t) = \frac{1}{A\sqrt{T}} f_n(t), \qquad n = 1, 2, 3$$

The signal vector representation is $\boldsymbol{s}_1 = (A\sqrt{T}, 0, 0), \boldsymbol{s}_2 = (0, A\sqrt{T}, 0)\boldsymbol{s}_3 = (0, 0, A\sqrt{T})$, and the signal constellation is 3D with a point on each axis.

For ML detection, the decision regions are obtained by drawing planes half way each pair of points.

(e) In order to express $x(t)$ as a linear combination of the functions, we need to calculate each of the projections onto the directions of $f_n(t)$

$$x_1 = \int x(t) f_1(t) dt = 0 \tag{5}$$

$$x_2 = \int x(t) f_2(t) dt = 0 \tag{6}$$

$$x_3 = \int x(t) f_3(t) dt = 0 \tag{7}$$

The signal is orthogonal to each of the functions and thus cannot be expressed as a linear combination.

(f) (i) Standard 3-FSK, hence K=3.

(ii) The vector representation is $\boldsymbol{s}_1 = (\sqrt{E_s}, 0, 0), \boldsymbol{s}_2 = (0, \sqrt{E_s}, 0)\boldsymbol{s}_3 = (0, 0, \sqrt{E_s})$ and the constellation is 3D as in part (d).

(iii) The optimum demodulator consists of a bank of correlators or a frequency down conversion, followed by a low-pass filter, a matched filter sampled at $t = mT$ and the detector. The optimal detector is MAP, which for equiprobable symbols is ML

$$\hat{m} = \arg\max_{i=1,\dots,3} P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{s}_i) = \arg\max_{i=1,\dots,3} y_i$$

$$\begin{aligned} P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{s}_i) &= P_{Y_1|X_1}(y_1|0)\cdots P_{Y_i|X_i}(y_i|\sqrt{E_\mathrm{s}})\cdots P_{Y_M|X_M}(y_M|0) && (8)\\ &= \frac{1}{(\sqrt{\pi N_0})^M}\, e^{-\frac{y_1^2}{N_0}}\cdots e^{-\frac{(y_i-\sqrt{E_\mathrm{s}})^2}{N_0}}\cdots e^{-\frac{y_M^2}{N_0}} && (9)\\ &= \frac{1}{(\sqrt{\pi N_0})^M}\, e^{-\frac{y_1^2+\cdots+y_M^2}{N_0}}\, e^{-\frac{E_\mathrm{s}}{N_0}}\, e^{\frac{2\sqrt{E_\mathrm{s}}y_i}{N_0}}. && (10)\end{aligned}$$

The first terms are all constants for the decision, and thus the only term remaining is the last one.

(iv) Using the union bound we have that for M-FSK

$$p_e \le (M-1)\mathbb{P}\big[N_2 - N_1 \ge \sqrt{E_\mathrm{s}}\big] = (M-1)Q\Bigg(\sqrt{\frac{E_s}{N_0}}\Bigg) = 2Q\Bigg(\sqrt{\frac{E_s}{N_0}}\Bigg)$$

4. (a) We have that

$$\begin{aligned} y(m) &= h_1u_1(m) + h_2u_2(m) + w(m) && (11)\\ y(m+1) &= -h_1u_2^*(m) + h_2u_1^*(m) + w(m+1) && (12)\\ y^*(m+1) &= -h_1^*u_2(m) + h_2^*u_1(m) + w^*(m+1) && (13)\end{aligned}$$

Rewriting the first and third equations in matrix form gives the result.

(b) We have that

$$\boldsymbol{H}^H\boldsymbol{H} = \begin{bmatrix} h_1^* & h_2 \\ h_2^* & -h_1 \end{bmatrix} \cdot \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} = \begin{bmatrix} |h_1|^2 + |h_2|^2 & 0 \\ 0 & |h_1|^2 + |h_2|^2 \end{bmatrix} \quad (14)$$

(c) If $\boldsymbol{H}$ is constant, then $\boldsymbol{H}^H\boldsymbol{w}$ is a filtered noise vector with zero mean as $\mathbb{E}[\boldsymbol{H}^H\boldsymbol{w}] = \boldsymbol{H}^H\mathbb{E}[\boldsymbol{w}] = \boldsymbol{0}$. For the covariance,

$$\mathbb{E}[\boldsymbol{H}^H\boldsymbol{w}\boldsymbol{w}^H\boldsymbol{H}] = \boldsymbol{H}^H\mathbb{E}[\boldsymbol{w}\boldsymbol{w}^H]\boldsymbol{H} = \boldsymbol{H}^H N_0\boldsymbol{I}\boldsymbol{H} = (|h_1|^2 + |h_2|^2)N_0\boldsymbol{I} \quad (15)$$

where the last step uses (b).

(d) From the previous sections, we know that the filtered noise vector $\boldsymbol{z} = \frac{1}{\sqrt{\gamma}}\boldsymbol{H}^H\boldsymbol{w}$ has zero mean and covariance $N_0\boldsymbol{I}$. Thus,

$$\boldsymbol{r} = \frac{1}{\sqrt{\gamma}}\boldsymbol{H}^H\boldsymbol{y} = \frac{1}{\sqrt{\gamma}}\boldsymbol{H}^H\boldsymbol{H}\boldsymbol{u} + \frac{1}{\sqrt{\gamma}}\boldsymbol{H}^H\boldsymbol{w} = \sqrt{\gamma}\boldsymbol{u} + \boldsymbol{z} \quad (16)$$

Therefore the ML detector is such that

$$\hat{\boldsymbol{u}} = \arg\min_{\boldsymbol{u}} \|\boldsymbol{r} - \sqrt{\gamma}\boldsymbol{u}\|^2 = \begin{bmatrix} \arg\min_{u_1} |r_1 - \sqrt{\gamma}u_1|^2 \\ \arg\min_{u_2} |r_2 - \sqrt{\gamma}u_2|^2 \end{bmatrix} \quad (17)$$

since the transformation decouples the two interfering channels.

(e) With 2 antennas, we observe a gain of $\gamma = 2$, hence we will have a 3 dB gain with respect to single antenna transmission with any modulation.

(f) We have that the error probability of each sub-channel is (for fixed h_1, h_2)

$$p_e = Q\left(\sqrt{\frac{2\gamma E_b}{N_0}}\right) \leq e^{-\gamma \frac{E_b}{N_0}}$$

Averaging over the fading we have that (with $\alpha_i = |h_i|^2$)

$$\begin{aligned}
p_e &\leq \mathbb{E}_\gamma[e^{-\gamma \frac{E_b}{N_0}}] && (18)\\
&= \int_0^\infty \int_0^\infty e^{-(\alpha_1+\alpha_2)\frac{E_b}{N_0}} e^{-\alpha_1} e^{-\alpha_2} d\alpha_1 d\alpha_2 && (19)\\
&= \left(\int_0^\infty e^{-\alpha_1(1+\frac{E_b}{N_0})} d\alpha_1\right) \cdot \left(\int_0^\infty e^{-\alpha_2(1+\frac{E_b}{N_0})} d\alpha_2\right) && (20)\\
&= \left(\frac{1}{1+\frac{E_b}{N_0}}\right)^2 && (21)
\end{aligned}$$

The diversity gain is thus 2.

(g) Having a number of receive antennas n_r would increase the diversity gain, which would become $2 \times n_r$.