A1

3F4 Solutions

1.) (a)(i) $S_x(\omega) = \lim_{T \to \infty} \dfrac{E[|X_T(\omega)|^2]}{T}$

The truncated signal $x_T(t)$ is

$$x_T(t) = \sum_{n=-N}^{N} a_n \delta(t - nT_s), \quad T = (2N+1)T_s$$

Now, $$X_T(\omega) = \int_{-\infty}^{\infty} x_T(t) e^{-j\omega t} dt$$

$$X_T(\omega) = \int_{-\infty}^{\infty} \sum_{n=-N}^{N} a_n \delta(t - nT_s) e^{-j\omega t} dt$$

$$= \sum_{n=-N}^{N} a_n \int_{-\infty}^{\infty} \delta(t - nT_s) e^{-j\omega t} dt$$

$$= \sum_{n=-N}^{N} a_n e^{-jn\omega T_s}$$

Now $$|X_T(\omega)|^2 = X_T(\omega) X_T^*(\omega)$$

$$= \sum_{n=-N}^{N} a_n e^{-jn\omega T_s} \sum_{k=-N}^{N} a_k e^{jk\omega T_s}$$

$$= \sum_{n=-N}^{N} \sum_{k=-N}^{N} a_n a_k e^{j(k-n)\omega T_s}$$

So, $$E[|X_T(\omega)|^2] = E\left[\sum_{n=-N}^{N} \sum_{k=-N}^{N} a_n a_k e^{j(k-n)\omega T_s}\right]$$

$$= \sum_{n=-N}^{N} \sum_{k=-N}^{N} E[a_n a_k] e^{j(k-n)\omega T_s}$$

now let $k = n + m$ and so $m = k - n$, so

$$E[|X_T(\omega)|^2] = \sum_{n=-N}^{N} \sum_{m=-N-n}^{N-n} E[a_n a_{n+m}] e^{j(n+m-n)\omega}$$

A2

$$E[|X_T(\omega)|^2] = \sum_{n=-N}^{N} \sum_{m=-N-n}^{N-n} R(m) e^{jm\omega T_s}$$

where $R(m) = E[a_n a_{n+m}]$.

Replace the outer sum over index n by $2N+1$,

$$E[|X_T(\omega)|^2] = (2N+1) \sum_{m=-N-n}^{N-n} R(m) e^{jm\omega T_s}$$

So,

$$S_x(\omega) = \lim_{N \to \infty} \frac{1}{(2N+1)T_s} (2N+1) \sum_{m=-N-n}^{N-n} R(m) e^{jm\omega T_s}$$

$$= \frac{1}{T_s} \sum_{m=-\infty}^{\infty} R(m) e^{jm\omega T_s}.$$

A.B.

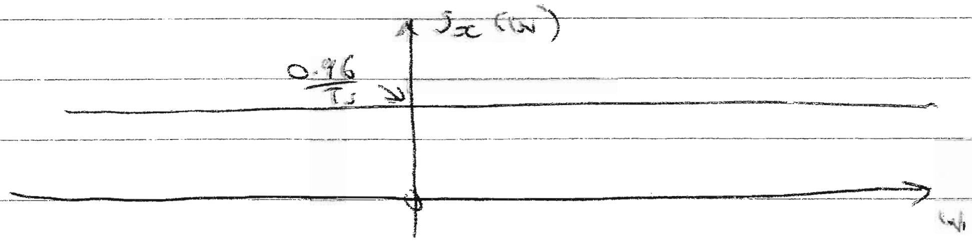
(ii) We need to calculate $R(m)$ for the (faulty) polar line coding scheme.

m	b_k	b_{k+m}	a_k	a_{k+m}	R_i	p_i	$R(m)$
0	0	0	−1.2	−1.2	1.44	0.4	0.96
	1	1	0.8	0.8	0.64	0.6	
1	0	0	−1.2	−1.2	1.44	$(0.4)^2 = 0.16$	
	0	1	−1.2	0.8	−0.96	0.4×0.6 0.24	0
	1	0	0.8	−1.2	−0.96	0.24	
	1	1	0.8	0.8	0.64	$(0.6)^2 = 0.36$	
2	0 (0)	0	−1.2	−1.2	1.44	0.064	
	0 (1)	0	−1.2	−1.2	1.44	0.096	
	0 (0)	1	−1.2	0.8	−0.96	0.096	
	0 (1)	1	−1.2	0.8	−0.96	0.144	
	1 (0)	0	0.8	−1.2	−0.96	0.096	0
	1 (1)	0	0.8	−1.2	−0.96	0.144	
	1 (0)	1	0.8	0.8	0.64	0.144	
	1 (1)	1	0.8	0.8	0.64	0.216	

Same result for $m \geq 2$, i.e., $R(m) = 0$.

So,

$$S_x(\omega) = \frac{1}{T_s} \times (0.96 \times 1) = \frac{0.96}{T_s}.$$



A24

(b) From the E and I databook,

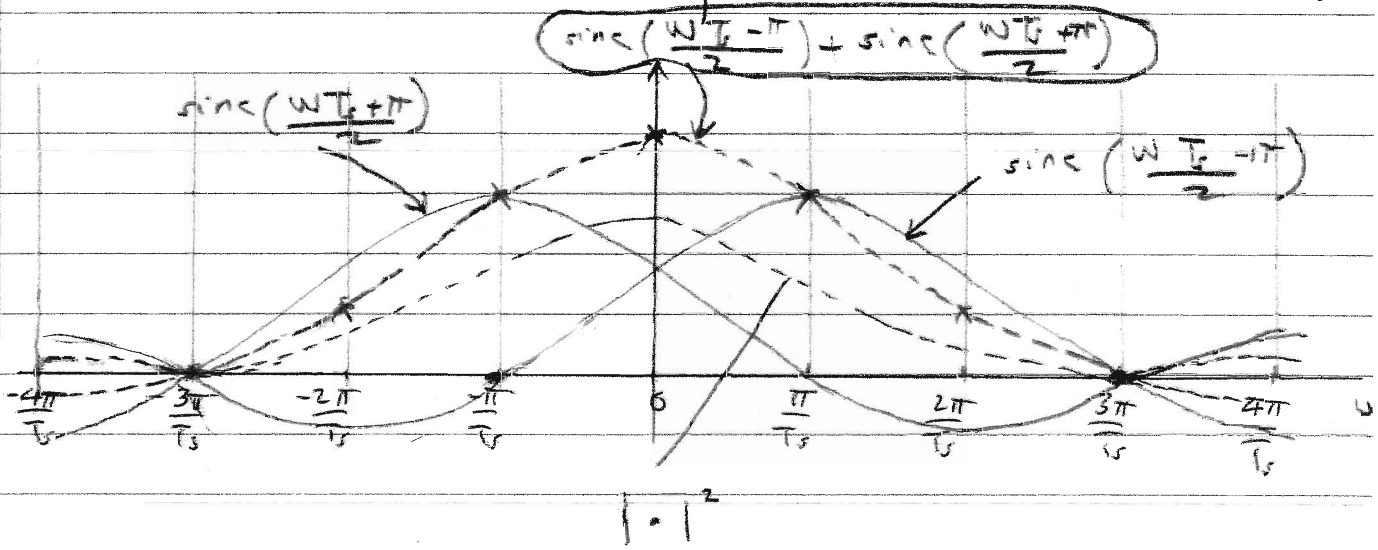
$$H(\omega) = \frac{T_s}{2}\left[\operatorname{sinc}\left(\frac{\omega T_s - \pi}{2}\right) + \operatorname{sinc}\left(\frac{\omega T_s + \pi}{2}\right)\right]$$

The transmitted PSD is given by,

$$S_y(\omega) = S_x(\omega)\,|H(\omega)|^2$$

$$= \frac{0.96}{T_s}\left|\frac{T_s}{2}\left[\operatorname{sinc}\left(\frac{\omega T_s - \pi}{2}\right) + \operatorname{sinc}\left(\frac{\omega T_s + \pi}{2}\right)\right]\right|^2$$

$$= 0.24\,T_s\left|\operatorname{sinc}\left(\frac{\omega T_s - \pi}{2}\right) + \operatorname{sinc}\left(\frac{\omega T_s + \pi}{2}\right)\right|^2$$



If rectangular pulses of duration $\pm T_s/2$ are used the resulting sinc frequency response has nulls at $\pm \frac{2\pi}{T_s}$.
Owing to the slow decay $\left(\sim \frac{1}{\omega^2}\right)$ of the sinc function the sidelobes of $S_y(\omega)$ will remain significant compared with those owing to the ½ sin pulse shape that decay quickly in excess of $\pm \frac{3\pi}{T_s}$.

We can show this as follows,

A5

Consider $H(\omega)$,

$$H(\omega) = \frac{T_s}{2}\left[\operatorname{sinc}\left(\frac{\omega T_s - \pi}{2}\right) + \operatorname{sinc}\left(\frac{\omega T_s + \pi}{2}\right)\right]$$

$$= T_s\left[\frac{\sin\left(\frac{\omega T_s}{2} - \frac{\pi}{2}\right)}{\omega T_s - \pi} + \frac{\sin\left(\frac{\omega T_s}{2} + \frac{\pi}{2}\right)}{\omega T_s + \pi}\right]$$

$$= T_s\left[\frac{\sin\frac{\omega T_s}{2}\cos\frac{\pi}{2} - \cos\frac{\omega T_s}{2}\sin\frac{\pi}{2}}{\omega T_s - \pi} + \frac{\sin\frac{\omega T_s}{2}\cos\frac{\pi}{2} + \cos\frac{\omega T_s}{2}\sin\frac{\pi}{2}}{\omega T_s + \pi}\right]$$

$$= T_s\left[\frac{-\cos\frac{\omega T_s}{2}}{\omega T_s - \pi} + \frac{\cos\frac{\omega T_s}{2}}{\omega T_s + \pi}\right]$$

$$= T_s\left[\frac{-(\omega T_s + \pi)\cos\left(\frac{\omega T_s}{2}\right) + (\omega T_s - \pi)\cos\left(\frac{\omega T_s}{2}\right)}{(\omega T_s - \pi)(\omega T_s + \pi)}\right]$$

$$= T_s\left[\frac{-\omega T_s\cos\left(\frac{\omega T_s}{2}\right) - \pi\cos\left(\frac{\omega T_s}{2}\right) + \omega T_s\cos\left(\frac{\omega T_s}{2}\right) - \pi\cos\left(\frac{\omega T_s}{2}\right)}{(\omega T_s - \pi)(\omega T_s + \pi)}\right]$$

$$= T_s\left[\frac{-2\pi\cos\left(\frac{\omega T_s}{2}\right)}{\omega^2 T_s^2 - \pi^2}\right]$$

$$= T_s \cdot \frac{2\pi}{\pi^2 - \omega^2 T_s^2}\cos\left(\frac{\omega T_s}{2}\right)$$

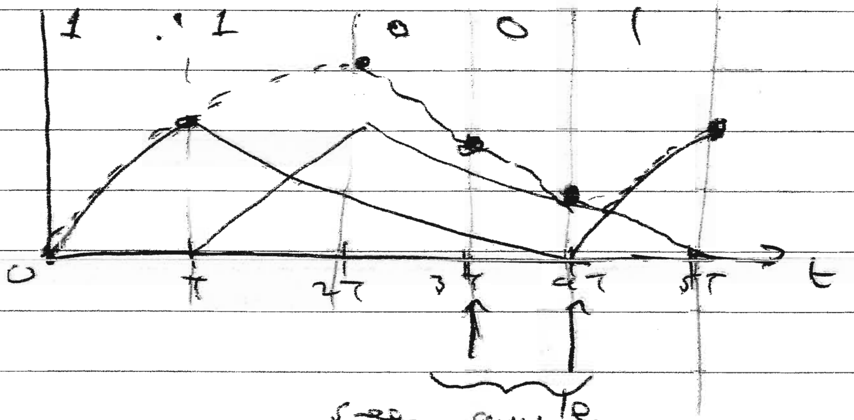
See $H(\omega)$ decays as $\frac{1}{\omega^2}$

$\therefore$ $|H(\omega)|^2$ " " $\frac{1}{\omega^4}$

compared with $1/\omega^2$ for rectangular pulses.

A∅

2) (a) If the system impulse response extends over more than one symbol duration then transmitted symbols become smeared in to adjacent symbol periods at the receiver. This effect can give rise to inter symbol interference (ISI) between the symbols that will impair bit error rate performance.



binary '1' sent as a pulse

binary '0' sent as 0V.

see sample values now far removed from ideal value of 0V ∴ decision errors more likely in the presence of additive noise.

A1

2) b)(i) Nyquists pulse shaping criterion is

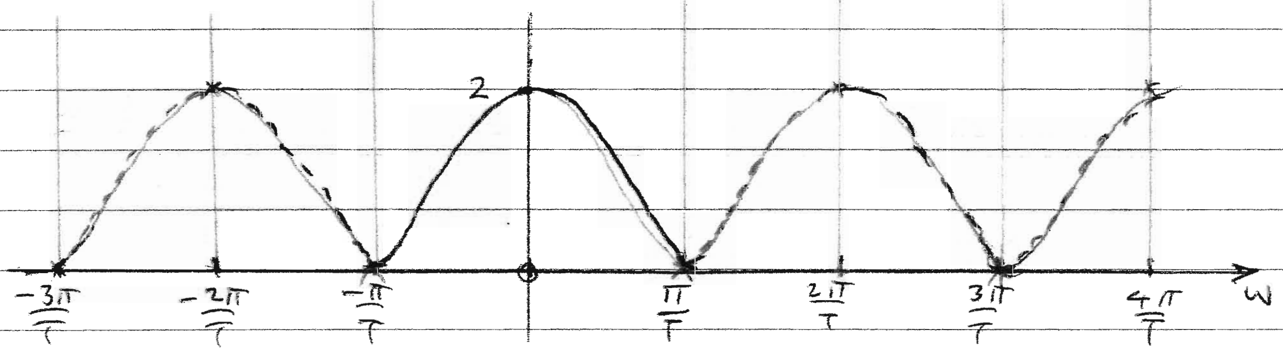
$$\sum_{k=-\infty}^{\infty} P_r\left(\omega - k\frac{2\pi}{T}\right) = T \qquad \text{or}$$

$$\sum_{k=-\infty}^{\infty} P_r\left(f - k/T\right) = T .$$

Now $e^{-j2\omega T}$ is equivalent to a delay of $2T$ in the time domain, hence will have no influence on the presence of ISI or otherwise, consequently we can concentrate on the $\left[1+\cos\left(\frac{\omega}{2B}\right)\right]$ term.

$$P_r'(\omega) = 1 + \cos\left(\frac{\omega}{2B}\right) \qquad |\omega| < 2\pi B$$

$$= 2\cos^2\left(\frac{\omega}{4B}\right) \qquad |\omega| < \frac{\pi}{T}$$



Now repeat $P_r'(\omega)$ at multiples of $\frac{2\pi}{T}$.

From above fig. we can see that

$$\sum P_r'\left(\omega - \frac{k\,2\pi}{T}\right) \neq \text{constant}.$$

So does not satisfy Nyquist pulse shaping criterion $\therefore$ will have ISI.

A2

b) (ii) We need to take the IFT of the pulse spectrum $P_r(\omega)$. Again we can initially ignore the $e^{-j2\omega T}$ term since this corresponds to a delay of $2T$ in the time domain.

$$p_r'(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_r'(\omega)\, e^{j\omega t}\, d\omega$$

$$p_r'(t) = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B}\left(1 + \cos\left(\frac{\omega}{2B}\right)\right) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B}\left(e^{j\omega t} + \frac{1}{2} e^{j\left(\frac{1}{2B}+t\right)\omega} + \frac{1}{2} e^{j\left(t-\frac{1}{2B}\right)\omega}\right) d\omega$$

$$= \frac{1}{2\pi}\left[\frac{1}{jt} e^{j\omega t} + \frac{1}{2j\left(t+\frac{1}{2B}\right)} e^{j\left(t+\frac{1}{2B}\right)\omega} + \frac{1}{2j\left(t-\frac{1}{2B}\right)} e^{j\left(t-\frac{1}{2B}\right)\omega}\right]_{-2\pi B}^{2\pi B}$$

$$= \frac{1}{2\pi}\left[\frac{2}{t}\sin 2\pi Bt + \frac{1}{\left(t+\frac{1}{2B}\right)}\sin\pi(2Bt+1) + \frac{1}{\left(t-\frac{1}{2B}\right)}\sin\pi(2Bt-1)\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{2}{t} - \frac{2t}{t^2-\left(\frac{1}{2B}\right)^2}\right)\sin 2\pi Bt\right]$$

$$= \frac{1}{2\pi}\left[\frac{2}{t}\left(\frac{1}{1-4B^2t^2}\right)\sin 2\pi Bt\right]$$

$$= \frac{2\pi B}{\pi}\left(\frac{1}{1-4B^2t^2}\right)\frac{\sin 2\pi Bt}{2\pi Bt}$$

$$= 2B\left(\frac{1}{1-4B^2t^2}\right)\operatorname{sinc} 2\pi Bt$$

(A3)

Taking into account the $e^{-j2\omega T}$ term yields,

$$p_r(t) = 2B\left(\frac{1}{1-4B^2(t-2T)^2}\right) \text{sinc}\, 2\pi B(t-2T$$

(iii) Ignoring delay, the time domain pulse is given by

$$p_r'(t) = \left(\frac{2B}{1-4B^2t^2}\right) \text{sinc}\, 2\pi Bt$$

substituting $B = \frac{1}{2T}$ yields,

$$p_r'(t) = \frac{1}{T}\left(\frac{1}{1-\frac{t^2}{T^2}}\right) \text{sinc}\, \frac{\pi}{T}t$$

First consider the sinc function.
This function has zero value when,

$$\frac{\pi}{T}t = \pm n\pi \quad , \quad n = 1, 2, \ldots \text{ etc.}$$

i.e, when $t = \frac{\pm n\pi T}{\pi} = \pm nT$.

The sinc function also has unity value at $t=0$

So $p_r'(t)$ at $t=0$ has value,

$$p_r'(0) = \frac{1}{T}.$$

However, for $t = \pm T$, $p_r'(t)$ becomes undefined, so we need to take limits of this function as $t \to$ (or $-T$), i.e,

$$p_r \lim_{t\to T} \frac{1}{T}\left(\frac{1}{1-\frac{t^2}{T^2}}\right) \frac{\sin \frac{\pi}{T}t}{\frac{\pi}{T}t}$$

$$\lim_{t \to T} \left(\frac{1}{\frac{T^2 - t^2}{T^2}} \right) \frac{\sin \frac{\pi}{T} t}{\pi t}$$

$$\lim_{t \to T} \left(\frac{T^2}{\pi (T^2 - t^2)} \right) \frac{\sin \frac{\pi}{T}}{t}$$

$$\lim_{t \to T} \frac{T^2 \sin \pi/_T t}{\pi T^2 t - \pi t^3}$$

Now apply l'Hopital's rule to give

$$\lim_{t \to T} \frac{T^2 \left(\frac{\pi}{T}\right) \cos \frac{\pi}{T} t}{\pi T^2 - 3\pi t^2} = \frac{-T\pi}{\pi T^2 - 3\pi T^2}$$

$$= \frac{-T\pi}{-2\pi T^2} = \frac{1}{2T}.$$

Now at $t = \pm nT$, $n = 2, 3 \ldots$

then

$$pr'(t) = 0 \quad \text{(owing to sin func)}.$$

So in the absence of delay at any time instant mT, the received sample value only depends on 3 values, i.e.,

$$r'(mT) = a_{m-1}\left(\frac{1}{2T}\right) + a_m \left(\frac{1}{T}\right) + a_{m+1} \left(\frac{1}{2T}\right)$$

i.e, the current symbol and the 2 adjacent symbols.

NOTE

$$r'(mT) = \sum_{k=-\infty}^{\infty} a_k \, pr'(mT - kT) = \sum_{k=-\infty}^{\infty} a_k \, pr'((m-k)T)$$

Only 3 non-zero values when $(m-k)T = -T, 0, T$

so $k = m+1$, m and $m-1$

$$r'(mT) = a_{m+1} \, pr'(-T) + a_m \, pr'(0) + a_{m-1} \, pr'(T)$$

3. (a) For arbitrary (analogue) modulation the transmitted waveform is

$$s(t) = a(t)\cos(\omega_c t + \phi(t))$$

where $a(t)$ & $\phi(t)$ are amplitude & phase modulation signals respectively. To combine these as phasor we rewrite this expression as

$$s(t) = Re\left[a(t)\, e^{j(\omega_c t + \phi(t))}\right]$$

$$= Re\left[p(t)\, e^{j\omega_c t}\right]$$

where $p(t) = a(t)\, e^{j\phi(t)}$

$p(t)$ is known as the phasor waveform.

For a digital modulation scheme such as BPSK, $p(t)$ is expressed as:

$$p(t) = e^{j\phi_0} \sum_k b_k\, g(t - kT_b) \qquad T_b = \text{bit period}$$

where $g(t)$ is a shaping pulse, usually rectangular, and ϕ_0 is an arbitrary phase shift due to path delays.

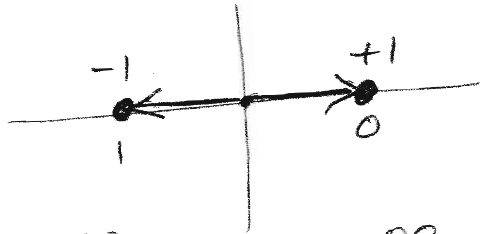
For more complex modulation schemes, the expression becomes

$$p(t) = e^{j\phi_0} \sum_k S_k\, g(t - kT_s) \qquad T_s = \text{symbol period}$$

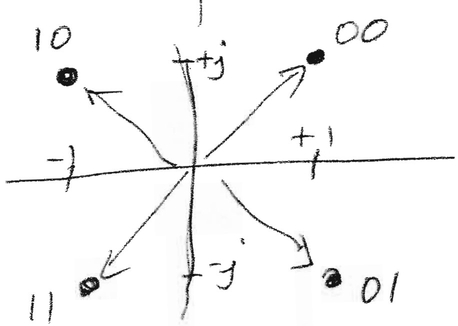
& the symbols S_k are complex functions of the bits b_n.

3. (b) Ignoring the phase offset ϕ_0, the constellations of phasors are:

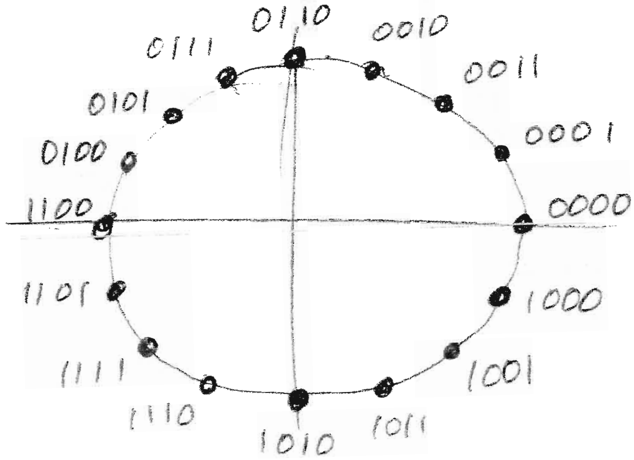
(i) BPSK:



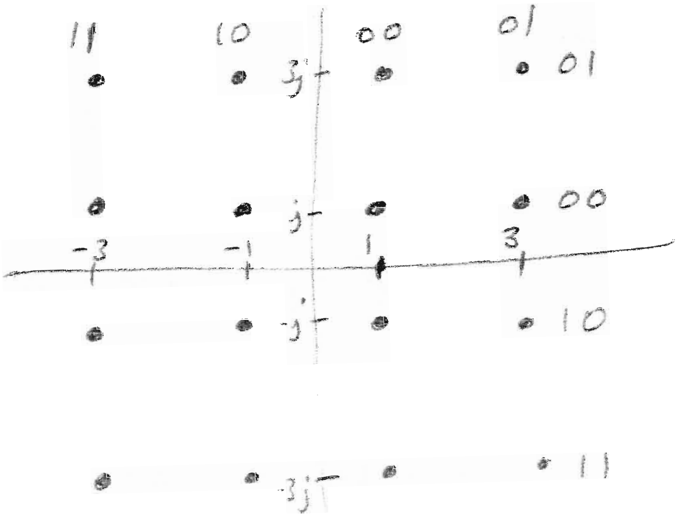
(ii) QPSK:



(iii) 16-PSK



(iv) 16-QAM



Phase offsets ϕ_0 will rotate each constellation by ϕ_0 ↺. The labelling of points is somewhat arbitrary but should follow unit distance (Gray) coding to minimise the decoded bit error rate.

3(c) The bandwidth efficiency is proportional to the number of information bits transmitted by each ~~sym~~ symbol. BPSK transmits only one bit per symbol and is the least efficient. QPSK transmits 2 bits/symbol and is twice as efficient as BPSK. 16-PSK & 16-QAM both transmit four bits per symbol ($\log_2 16 = 4$), and are 4 times as efficient as BPSK.

The performance in noise is a function of the separation of the constellation points, for ~~g~~ a given energy per bit E_b in the constellation. BPSK & QPSK both have the same separation of near-neighbours, for constant E_b, ~~~~ and will have the best performance against noise. 16-PSK will have the smallest separation and ~~be the~~ have the worst performance of the four schemes against noise. 16-QAM is somewhat better than 16-PSK because it distributes the 16 states equally over the 2-D plane, rather than constraining the point to be on a circle.

Implementation complexity is a function of the number of states being detected. BPSK is simplest, QPSK is twice the complexity of BPSK, 16-QAM is about twice the complexity of QPSK because 4-states are detected along each of the two axes. 16-PSK is

3(c) (cont.) the most complex because 16 phase states must be detected individually.

3(d) Absolute phase is usually difficult to determine because of uncertainties over absolute time-of-day and the precise path delay from transmitter to receiver. These would need to be known to an accuracy of $\ll$ 1 period of the radio carrier wave.

Differential measurement of phase is usually the method employed to overcome this. In BPSK + QPSK, differential coding of the data onto the constellation phase states is very effective. The receiver must "phase-lock" to the received signal phase states and then differential decoding of the detected data bits can recover the original data with no ambiguities. The only penalty is a doubling of the detected bit error probability, but this can be compensated by a small (~ 0.5 dB) increase in signal power. Differential decoding will also work with 16-PSK and 16-QAM, although the lack of circular symmetry in QAM makes this quite tricky to achieve. Often, a small proportion of reference ~~[illegible]~~ symbols are used with QAM instead.

4. (a) OFDM is a technique for reducing the symbol rate of a digital signal by splitting a single high-speed ~~data~~ bit stream into many (N) separate bit streams, each of which is modulated onto a separate sub-carrier wave. The sub-carriers are spaced apart by ~~the~~ a frequency equal to the sub-carrier symbol modulation rate, so that all ~~the~~ N sub-carriers are orthogonal to each other when analysed over the symbol period. Hence no inter-carrier interference is generated. The Fourier transform has this orthogonality property and hence the OFDM signal may be detected efficiently by the Fast Fourier Transform (FFT) and may be generated efficiently by the inverse FFT.

(b) The main problem that OFDM overcomes is inter-symbol-interference (ISI) caused by different path delays from transmitter to receiver. When the differential delay is greater than the transmitted symbol period, nearby symbols will arrive at the receiver simultaneously & not be distinguishable from each other. OFDM overcomes this by dramatically reducing the

4.(b)(cont) symbol rate, to $\frac{1}{N}$ of what it would be with a single subcarrier, so that the symbol period on each subcarrier is much longer than any expected differential path delays. Guard bands are also introduced between symbols ~~to~~ so that orthogonality of the received sub-carriers is preserved, even when there are differential delays up to the duration of the guard band.

Coding is needed with OFDM, because multipath transmission can cause frequency-selective fading, which will suppress a small proportion of the transmitted subcarriers at the receiver. Error-correction coding is very effective at recovering the complete data-stream, even when a small proportion of the bits are corrupted by selective fading.

(c) Total TV bit rate $= 3 \times 6 + 4 \times 2$

$= 18 + 8 = 26$ Mbit/s.

After error coding, the bit rate becomes $26 \times \frac{3}{2} = 39$ Mbit/s

The subcarriers are 4 kHz apart, so the analysis period for orthogonality at the receiver is $\frac{1}{4000}$ s $= 250\mu s$.

4(c) (cont.)

The guard period must be equivalent to 3 km at the ~~a~~ velocity of light (3.10^8 m/s) $= \frac{3000}{3.10^8} = 10^{-5}$ s.

Hence $T_{analysis} + T_{guard} = 250 + 10 = 260\ \mu s$.

$\therefore$ With 64-QAM (6 bits per symbol), the bit rate per subcarrier ~~bit~~ $= \frac{6}{260.10^{-6}} = 23077$ bit/s.

$\therefore$ No of <u>active</u> subcarriers $= \frac{39.10^6}{23077} = 1690$

Allowing 10% of carriers to be (non-active) pilot tones, the total number of subcarriers $= \frac{1690}{0.9} = \underline{\underline{1878}}$

The subcarrier spacing is 4 kHz, so the bandwidth needed is approx $(1878 + 1).4000 = \underline{\underline{7.516\text{ MHz}}}$.

In practise, we would need to allow ~~a lo~~ more bandwidth than this to avoid interference from ~~adjact~~ adjacent OFDM signals, so $\underline{\underline{8\text{ MHz}}}$ would be a safer bandwidth to use.