3F4 Data Transmission
Engineering Tripos 2020/21 – Solutions

Question 1

(a) i) The first basis function can be chosen as $s_1(t)$ scaled to have unit energy, and we use Gram-Schmidt to find the second basis function: [30%]

$$\phi_1(t) = \frac{1}{\sqrt{E}} s_1(t), \quad \phi_2(t) = \frac{s_{2\perp}(t)}{\|s_{2\perp}(t)\|},$$

where

$$\begin{aligned} s_{2\perp}(t) &= s_2(t) - \langle \phi_1, s_2 \rangle \phi_1(t) = s_2(t) + \frac{\sqrt{E}}{2}\phi_1(t) \\ &= \begin{cases} \frac{1}{2}\sqrt{\frac{E}{2T}}, & 0 \le t \le 2T, \\ -\sqrt{\frac{E}{2T}}, & 2T < t \le 3T. \end{cases} \end{aligned}$$

Since $\|s_{2\perp}(t)\| = \sqrt{3E}/2$, we have

$$\phi_2(t) = \begin{cases} \sqrt{\frac{1}{6T}}, & 0 \le t \le 2T, \\ -\sqrt{\frac{2}{3T}}, & 2T < t \le 3T. \end{cases}$$

Using these basis functions, we have $s_1(t) = \sqrt{E}\,\phi_1(t)$, and $s_2(t) = -\sqrt{E/4}\,\phi_1(t) + \sqrt{3E/4}\,\phi_2(t)$

ii) In the two-dimensional signal space, the signal vectors are:

$$\underline{s}_1 = [\sqrt{E}, 0], \quad \underline{s}_2 = [-\sqrt{E/4},\ \sqrt{3E/4}].$$

The optimal receiver first computes $\underline{r} = [r_1, r_2]$, where

$$r_1 = \langle y(t),\ \phi_1(t) \rangle, \quad r_2 = \langle y(t),\ \phi_2(t) \rangle.$$

The optimal detector then uses the minimum-distance rule: [20%]

$$\underline{\hat{x}} = \begin{cases} \underline{s}_1, & \text{if } \|\underline{r} - \underline{s}_1\|^2 \le \|\underline{r} - \underline{s}_2\|^2, \\ \underline{s}_2, & \text{otherwise.} \end{cases}$$

Note: The choice of orthonormal basis in part (i) is not unique. Each choice will correspond to a different pair $(\underline{s}_1, \underline{s}_2)$, each of these pairs will have the same value for the distance $\|\underline{s}_1, \underline{s}_2\|$.

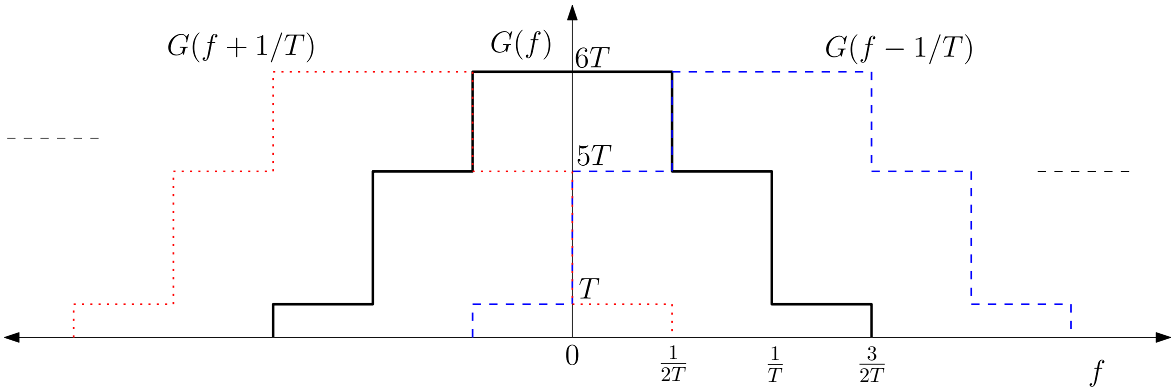
iii) If message $i \in \{1, 2\}$ is transmitted,

$$r_1 = s_{i,1} + n_1, \quad r_2 = s_{i,2} + n_2.$$

The decision boundary is a line perpendicular to $(\underline{s}_1 - \underline{s}_2)$. Detection is affected only by the noise component in the direction $(\underline{s}_1 - \underline{s}_2)$. Since the AWGN noise is rotationally invariant, the noise component in the direction is $\mathcal{N}(0, N_0/2)$. Denoting this random variable by N', the probability of error is [20%]

$$P_e = P\left(N' > \frac{\|\underline{s}_1 - \underline{s}_2\|}{2}\right) = P\left(N' > \frac{1}{2}\sqrt{3E}\right) = \mathcal{Q}\left(\sqrt{\frac{3E}{2N_0}}\right).$$

(b) Since $p(t)$ is real and even, so is $P(f)$. We have $g(t) = p(t) \star p(-t) = p(t) \star p(t)$, which implies $G(f) = P(f)^2$. We plot $G(f), G(f-1/T)$ and $G(f+1/T)$ in the figure below. [30%]



We see that

$$\sum_{n=-\infty}^{\infty} G\left(f - \frac{n}{T}\right) = 6T + 5T + T = 12T. \tag{1}$$

The Nyquist pulse criterion says that id $\sum_n G\left(f - \frac{n}{T}\right) = T$, then $g(nT) = 1$ for $n = 0$, and 0 for other integers n. The RHS in (1) is scaled by a factor of 12, therefore $g(nT) = 12$ for $n = 0$, and 0 otherwise.

Therefore, from the result in (ii), the filter output is

$$r(nT) = X_n g(nT) = \begin{cases} 12X_n, & n = 0 \\ 0, & n = \pm 1, \pm 2, \ldots \end{cases}$$

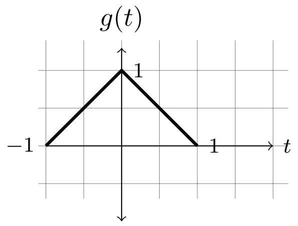
Assessor's comment: Some got into a tangle in the Gram-Schmidt for part (a).(i) (possibly due to time pressure) as the calculation was a bit involved; others erroneously gave three basis functions, or just scaled the given functions to have unit energy. Most did well on parts (a).(ii) and (a).(iii), which asked for the optimal receiver and its error probability. Part (b) on Nyquist pulse criterion had several good answers, but some tried to do it via time domain, which is more time consuming.

Question 2

(a) i) The channel output $y(t) = \sum_k X_k[p(t-k) - 0.4p(t-k-0.5) + 0.1p(t-k-1.5)] + n(t)$. Passing this through a filter with impulse response $p(-t)$, we obtain [10%]

$$\begin{aligned} r(t) &= \sum_k X_k[p(t-k) - 0.4p(t-k-0.5) + 0.1p(t-k-1.5)] \star p(-t) + n(t) \star p(-t) \\ &= \sum_k X_k[g(t-k) - 0.4g(t-k-0.5) + 0.1g(t-k-1.5)] + \tilde{n}(t) \end{aligned}$$

ii) The pulse $g(t) = p(t) \star p(-t) = \int_0^t p(\tau)p(\tau - t)d\tau$ is: [25%]



Since $g(t)$ is non-zero only for $-1 < t < 1$, the sampled filter output is

$$\begin{aligned} r(m) &= \sum_k X_k[g(m-k) - 0.4g(m-k-0.5) + 0.1g(m-k-1.5)] + \tilde{n}(m) \\ &= X_m[g(0) - 0.4g(-0.5)] + X_{m-1}[-0.4g(0.5) + 0.1g(-0.5)] + X_{m-2}[0.1g(0.5)] + \tilde{n}(m) \\ &= 0.8X_m - 0.15\, X_{m-1} + 0.05X_{m-2} + \tilde{n}(m) \end{aligned}$$

iii) The channel impulse response is $\underline{g} = [g_0 = 0.8,\ g_1 = -0.15,\ g_2 = 0.05]$. Let the 3-tap filter be $\underline{h} = [h_0, h_1, h_2]$. Then, the filter output for $m \geq 0$ is [25%]

$$y_m = X_m f_0 + \sum_{j=1}^{4} X_{m-j} f_j + \sum_{i=0}^{2} h_i n_{m-i},$$

where $f_j = \sum_{i=0}^{2} h_i g_{j-i}$.

To obtain the three tap ZF equalizer we solve:

$$\begin{aligned} &f_0 = h_0 g_0 = 1 \quad \Rightarrow \quad h_0 = 1/0.8 = 1.25, \\ &f_1 = h_0 g_1 + h_1 g_0 = 0 \quad \Rightarrow \quad h_1 = -h_0 g_1 / g_0 = 0.2344, \\ &f_2 = h_0 g_2 + h_1 g_1 + h_2 g_0 = 0 \quad \Rightarrow \quad h_2 = -(h_0 g_2 + h_1 g_1)/g_0 = -0.0342 \end{aligned}$$

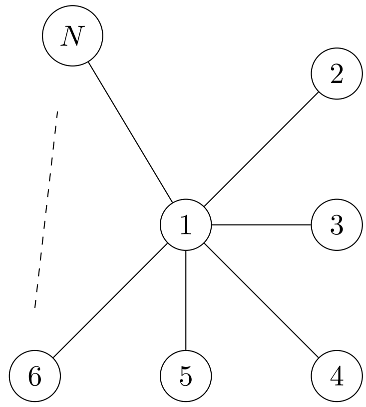
With this choice, we have $f_3 = h_1 g_2 + h_2 g_1 = 0.0168$ and $f_4 = h_2 g_2 = -0.0017$.

Therefore, the filter output is

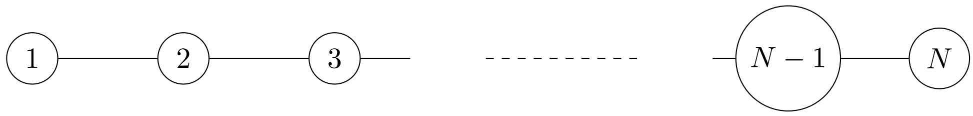
$$Y_m = X_m + \underbrace{0.0168_{m-3} - 0.0017X_{m-4}}_{\text{residual interference}} + \underbrace{\sum_{i=0}^{2} h_i n_{m-i}}_{\text{additive noise}}.$$

iv) The noise enhancement factor is $(h_0^2 + h_1^2 + h_2^2) = 1.619$. The MMSE equaliser achieves the optimal tradeoff between residual interference and noise enhancement by minimizing the expected squared error between X_m and the equaliser estimate $\hat{X}_m$. [10%]

(b) i) For the Bellman-Ford algorithm will find all minimum cost paths within two iterations, each node should be at most two links away from every other node. For large N, this can be achieved by the following 'hub and spoke' configuration: [15%]



(c) ii) The Bellman-Ford algorithm will be guaranteed to take $(N-1)$ iterations if there is at least one pair of nodes for which every path between them has $(N-1)$ links. This can be achieved (for example) by arranging the nodes in a line configuration, so that the nodes at the two ends are separated by $(N-1)$ links. [15%]



Note: Other valid configurations may be possible for (b).(i) and (b.(ii).

Assessor's comment: Part (a).(i) was done well by most. Many made mistakes in part (a).(ii) figuring out which samples of $g(\cdot)$ played a role in determining the ISI for $r(m)$. For part (b).(i), a common mistake was giving a fully connected network as an example for which the algorithm always converges in two steps. Note that a (long) indirect path could be the one with minimum cost even in a fully connected network, and this could potentially take more than two iterations to find. Some gave examples that worked only for a specific, small value of N, though the question specifies that N is large.

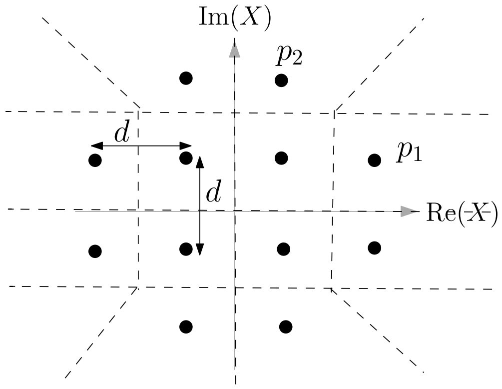
Question 3

(a) i) The squared magnitude of each of the four inner symbols is $(d/2)^2 + (d/2)^2 = d^2/2$. For each of the eight outer symbols, the squared magnitude is $(d/2)^2 + (3d/2)^2 = 5d^2/2$. Therefore the average symbol energy is [15%]

$$E_s = \frac{1}{12}\left(4 \cdot \frac{d^2}{2} + 8 \cdot \frac{5d^2}{2}\right) = \frac{11}{6}d^2.$$

Since each symbol carries $\log_2 12$ bits, the average energy per symbol is $E_b = \frac{11}{6\log 12}d^2$.

ii) The decision boundaries are as shown in the figure below. [10%]



iii) The outer symbols will all have the same probability of detection error, denoted by $P_{e,outer}$. Similarly, the inner ones will have the same probability of detection error, denoted by $P_{e,inner}$. First consider an inner symbol, say $(d/2, d/2)$. With $(N_R, N_I) \sim_{\text{iid}} \mathcal{N}(0, N_0/2)$, we have [25%]

$$\begin{aligned} P_{\text{e,inner}} &= P\left(\left\{\frac{d}{2} + N_R < 0\right\} \cup \left\{\frac{d}{2} + N_R > d\right\} \cup \left\{d < \frac{d}{2} + N_I < 0\right\} \cup \left\{\frac{d}{2} + N_I > d\right\}\right) \\ &\leq P(N_R < -d/2) + P(N_R > d/2) + P(N_I < -d/2) + P(N_I > d/2) = 4\mathcal{Q}\left(\sqrt{d^2/(2N_0)}\right). \end{aligned}$$

The last equality is obtained by standardising each of the normals by dividing both sides of the inequality by $\sqrt{N_0/2}$.

We now bound $P_{\text{e,outer}}$ by considering the outer symbol $p_1 = (3d/2, d/2)$. Denoting by $\tilde{N}$ the component of the noise in the direction from p_1 to p_2, we have

$$\begin{aligned} P_{\text{e,outer}} &= P\left(\left\{N_R < -d/2\right\} \cup \left\{N_I < -d/2\right\} \cup \left\{\tilde{N} > \sqrt{2}d/2\right\}\right) \\ &\leq P(N_R < -d/2) + P(N_I < -d/2) + P(\tilde{N} > \sqrt{2}d/2). \end{aligned}$$

Due to the rotational invariance of Gaussian $\tilde{N} \sim \mathcal{N}(0, N_0/2)$ (see Examples Paper 2, where this is shown by expressing $\tilde{N}$ as a linear combination of N_I and N_R). Standardising each of the normals by dividing $\sqrt{N_0/2}$, we therefore obtain

$$P_{\text{e,outer}} \leq 2\mathcal{Q}\left(\sqrt{d^2/(2N_0)}\right) + \mathcal{Q}\left(\sqrt{d^2/N_0}\right).$$

Combining the bounds above, we can bound the overall probability of error as

$$\begin{aligned} P_e &= \frac{8}{12}P_{\text{e,outer}} + \frac{4}{12}P_{\text{e,inner}} \leq \frac{8}{3}\mathcal{Q}\left(\sqrt{d^2/(2N_0)}\right) + \frac{2}{3}\mathcal{Q}\left(\sqrt{d^2/(N_0)}\right) \\ &= \frac{8}{3}\mathcal{Q}\left(\frac{3\log 12}{11}\frac{E_b}{N_0}\right) + \frac{2}{3}\mathcal{Q}\left(\frac{6\log 12}{11}\frac{E_b}{N_0}\right) \end{aligned}$$

For the last equality, we have expressed d^2 in terms of E_b using $E_b = \frac{11}{6\log 12}d^2$.

(b) i) The optimal decision rule for X given $Y = y$ is the MAP decision rule, which in this case is: decode 0 if $P(X = 0)P_0(y) > P(X = 1)P_1(y)$, and decode 1 otherwise. [20%]

$$\hat{X}_{MAP}(y) = \begin{cases} 0, & \text{if } (1-p)\cdot \lambda_0^y e^{-\lambda_0} \geq p \cdot \lambda_1^y e^{-\lambda_1} \\ 1, & \text{otherwise.} \end{cases}$$

Simplyifying, we obtain

$$\hat{X}_{MAP}(y) = \begin{cases} 0, & \text{if } 0 \leq y \leq \frac{\lambda_1 - \lambda_0 + \ln \frac{1-p}{p}}{\ln \frac{\lambda_1}{\lambda_0}}, \\ 1, & \text{otherwise.} \end{cases}$$

The threshold $T = \frac{\lambda_1 - \lambda_0 + \ln \frac{1-p}{p}}{\ln \frac{\lambda_1}{\lambda_0}}$.

[Since y is a positive integer, the decision rule will decode 0 for $0 \leq y \leq \lfloor T \rfloor$, and decode 1 for $y \geq \lfloor T \rfloor + 1$.]

ii) For $\lambda_1 = 10, \lambda_0 = 2$ and $p = (1-p)$, we have $T = \frac{8}{\ln 5} = 4.9707$. Since y is an integer, the decision rule reduces to [10%]

$$\hat{X}_{MAP}(y) = \begin{cases} 0, & \text{if } 0 \leq y \leq 4 \\ 1, & y \geq 5. \end{cases}$$

iii) The probability of error is [20%]

$$\begin{aligned} P_e &= P(X = 0)P(Y > T \mid X = 0) + P(X = 1)P(Y \leq T \mid X = 1) \\ &= (1-p)e^{-\lambda_0} \sum_{r=\lfloor T \rfloor + 1}^{\infty} \frac{\lambda_0^r}{r!} + pe^{-\lambda_1} \sum_{r=0}^{\lfloor T \rfloor} \frac{\lambda_1^r}{r!} \end{aligned}$$

Assessor's comment: In part (a).(ii), some made mistakes in sketching the optimal decision boundaries for the outer points. Many also gave rather loose (or sometimes, incorrect) upper bounds for the error probability in (a).(iii); a common mistake was in analysing the error probability for an outer point due to the neighbouring constellation point on the diagonal. Part (b) on MAP detection over a Poisson channel was generally well done.

Question 4

(a) The number of subcarriers $N = \frac{\text{bandwidth}}{\text{spacing}} = \frac{10\times 10^6}{4000} = 2500$.

Each OFDM block is transmitted in $T+\Delta$ seconds, where $T = NT_s = \frac{1}{\text{spacing}} = 250\mu$s, and the guard interval $\Delta = 10\mu$s. [20%]

The number of subcarriers carrying user information is $0.9N$. Each 16-QAM symbol corresponds to 4 bits, and there is a rate 1/2 error correcting code. Hence

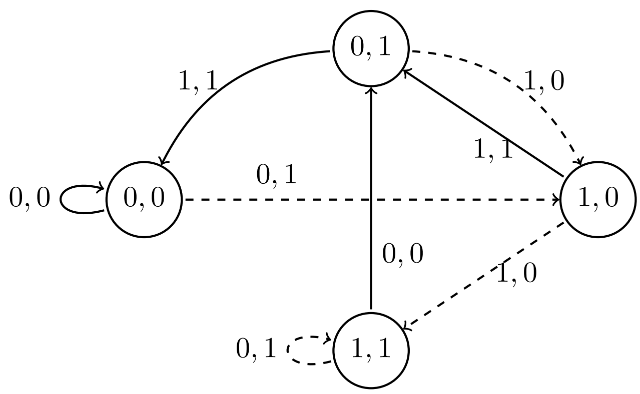
$$R_{user} = \frac{0.9N \cdot 4 \cdot 1/2}{(T+\Delta)} = 17.308 \times 10^6 \text{ bits per s.}$$

Denoting the length of the cyclic prefix by L, the length of the guard interval $\Delta = LT_s$, where $T_s = \frac{250}{2500} = 0.1\mu$s. Therefore

$$L = \frac{\Delta}{T_s} = 100.$$

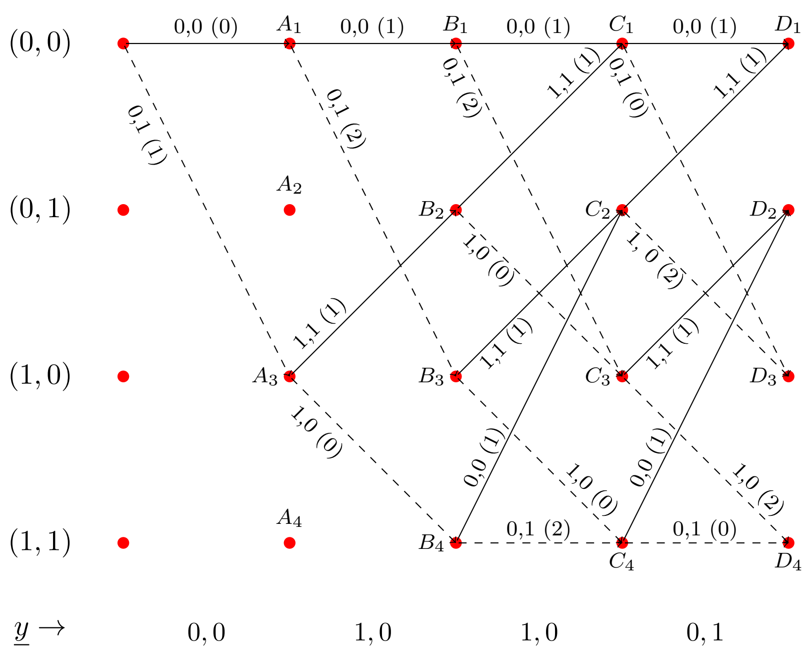
(b) i) The rate of the code is 1/2, as two coded bits are produced for each input bit. [5%]

ii) The state diagram is shown below, with a transitions corresponding to 0 inputs by solid lines, and transitions corresponding to 1 inputs by dashed lines. [20%]



ii)

The following trellis diagram shows the code bits along each transition, along with the Hamming distance from the corresponding bits in $\underline{y}$



The cumulative distance denoted by $d(\cdot,\cdot)$ from the origin to the nodes in each stage: [45%]

Stage A) $d(00,A_1)=0,\ d(00,A_3)=1$
Stage B) $d(00,B_1)=1,\ d(00,B_2)=2,\ d(00,B_3)=2,\ d(00,B_4)=1$
Stage C) $d(00,C_1)=2(\text{via } B_1),\ d(00,C_2)=2(\text{via } B_4),\ d(00,C_3)=2(\text{via } B_2),\ d(00,C_4)=2\,(\text{via } B_3)$
Stage D) $d(00,D_1)=3(\text{via } C_1 \text{ or } C_2),\ d(00,D_2)=3(\text{via } C_3 \text{ or } C_4),\ d(00,D_3)=2(\text{via } C_1),\ d(00,D_4)=2(\text{via } C_4)$

There are two distinct paths each of whose codewords have distance 2 from $\underline{y}$:

$$00 - A_1 - B_1 - C_1 - D_3 \leftrightarrow \underline{\hat{x}}_1 = 00\,00\,00\,01, \quad \underline{\hat{s}}_1 = 0001,$$
$$00 - A_1 - B_3 - C_4 - D_4 \leftrightarrow \underline{\hat{x}}_2 = 00\,01\,10\,01, \quad \underline{\hat{s}}_2 = 0111,$$

(c) There are two codewords which are at a distance 2 from $\underline{y}$ and no other codeword with a smaller distance. Since the Viterbi algorithm determines a codeword closest in Hamming distance to $\underline{y}$, both of these are equally valid outputs of the decoder. [10%]

Assessor's comment: The most popular question in the paper. Some candidates used the wrong number of subcarriers to determine the OFDM symbol period in part (a), but both parts were generally well-answered by most.