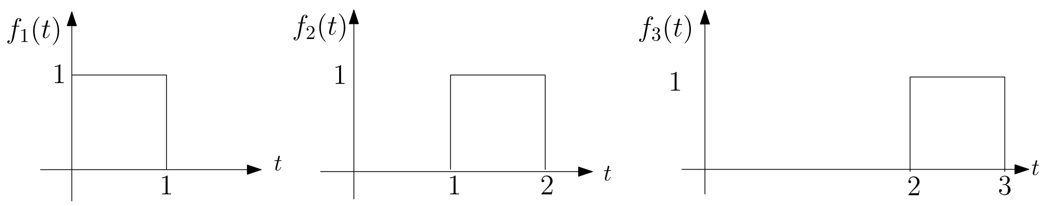
# 3F4 Data Transmission
# Engineering Tripos 2017/18 – Solutions

## Question 1

(a) i) An orthonormal basis for the set of waveforms is given below. [10%]




ii) Each waveform $s_i(t)$, for $i \in \{1, 2, 3, 4\}$, can be expressed as $s_i(t) = s_{i,1}f_1(t) + s_{i,2}f_2(t) + s_{i,3}f_3(t)$, where the coefficients are calculated as

$$s_{i,1} = \int_{\mathbb{R}} s_i(t)f_1(t)dt, \quad s_{i,2} = \int_{\mathbb{R}} s_i(t)f_2(t)dt, \quad s_{i,3} = \int_{\mathbb{R}} s_i(t)f_3(t)dt.$$

Using this, we obtain:

$$\begin{aligned} s_1(t) &= 2\, f_1(t) + 2\, f_2(t) + 2\, f_3(t), \\ s_2(t) &= 2\, f_1(t) + 0\, f_2(t) + 0\, f_3(t), \\ s_3(t) &= 0\, f_1(t) - 2\, f_2(t) - 2\, f_3(t), \\ s_4(t) &= 2\, f_1(t) + 2\, f_2(t) + 0\, f_3(t). \end{aligned}$$

iii) The receiver first computes inner product of $y(t)$ with each of the basis functions: [25%]

$$r_1 = \int_{\mathbb{R}} y(t)f_1(t)dt, \quad r_2 = \int_{\mathbb{R}} y(t)f_2(t)dt, \quad r_3 = \int_{\mathbb{R}} y(t)f_3(t)dt.$$

If $x(t) = s_i(t)$, then the vector $\underline{r} = [r_1, r_2, r_3]$ is given by

$$\underline{r} = \underline{s}_i + \underline{n},$$

where $\underline{s}_i = [s_{i,1}, s_{i,2}, s_{i,3}]$, and $\underline{n} = [n_1, n_2, n_3]$. Here

$$n_1 = \int_{\mathbb{R}} n(t)f_1(t), \quad n_2 = \int_{\mathbb{R}} n(t)f_2(t), \quad n_3 = \int_{\mathbb{R}} n(t)f_3(t),$$

are zero-mean i.i.d. Gaussian random variables. (The variance is determined by the power spectral density of $n(t)$.)

Since the messages are equally likely, the optimal decision rule is ML or minimum-distance decoding rule:

$$\hat{s} = \underset{i \in \{1,2,3,4\}}{\arg\min} \ \|\underline{r} - \underline{s}_i\|^2.$$

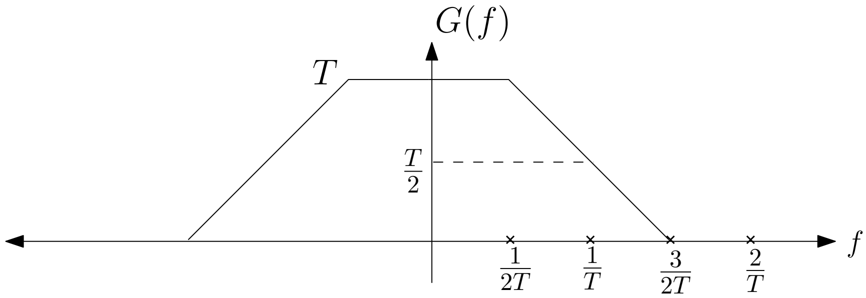
(b) i) The filter output is given by [10%]

$$r(t) = x(t) \star q(t) = \sum_k X_k p(t - kT) \star q(t)$$
$$= \sum_k X_k \int_{\mathbb{R}} q(u)p(t - kT - u)du = \sum_k X_k g(t - kT)$$

because $g(t) = \int_{\mathbb{R}} q(t)p(t-u)du$.

ii) If $\sum_{k=-\infty}^{\infty} G(f - \frac{k}{T}) =$ constant, then there will be no ISI. If the constant on the RHS equals $T$, then the sampled output at time $nT$ exactly equals $X_n$, for $n \in \mathbb{Z}$. [10%]

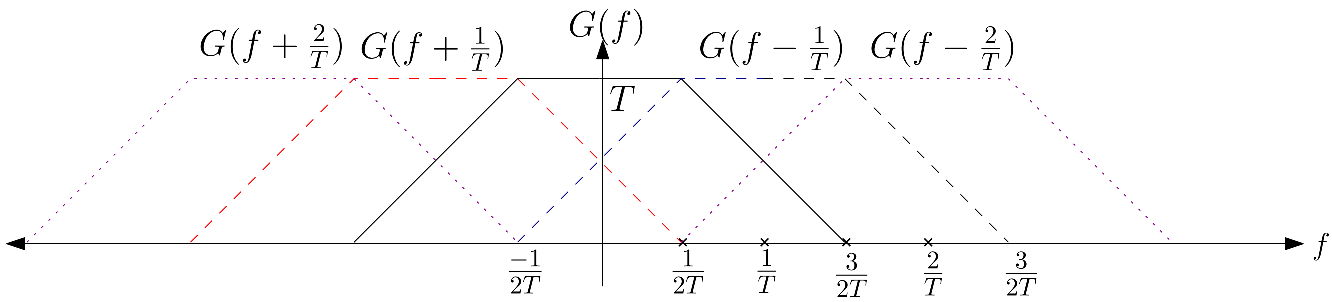
iii) Noting that $G(f) = P(f)Q(f)$, we have [10%]




iv) We show below $G(f)$ superimposed with the shifted versions $G(f - \frac{1}{T})$ and $G(f + \frac{1}{T})$. We see that [20%]

$$\sum_{k=-\infty}^{\infty} G\left(f - \frac{k}{T}\right) = 2T.$$

Therefore the sampled output at time $nT$ equals $2X_n$, for $n = 0, 1, 2, \ldots$




***Assessor's comment:*** *For the last part of the question, many tried to determine $r(nT)$ via the time-domain expression for $r(t)$, which can be complicated to compute. Many also made mistakes in showing b.i) via a change of variables in the convolution integral.*

## Question 2

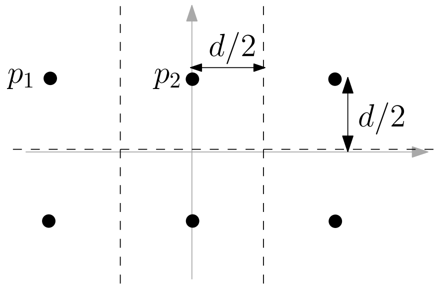
(a) i) The corner points have energy $d^2 + (d/2)^2 = \frac{5d^2}{4}$, and the two middle ones have energy $d^2/4$. Therefore the average energy per symbol is [15%]

$$E_s = \frac{1}{6}\left[4 \cdot \frac{5d^2}{4} + 2 \cdot \frac{d^2}{4}\right] = \frac{11d^2}{12}.$$

The average energy per bit is

$$E_b = \frac{E_s}{\log_2 6} = \frac{11d^2}{12\log_2 6}.$$

ii) The decision regions are as shown below, with the dashed lines representing the decision boundaries.




iii) The detector attempts to recover the transmitted symbol $X$ from $Y = X + N$, where the the real and imaginary parts of $N$ are i.i.d Gaussian $\sim \mathcal{N}(0, N_0/2)$. $X$ is drawn uniformly from the 6-QAM constellation. [30%]

When $X$ is one of the four corner points, say $p_1$, the prob. of error is

$$\begin{aligned}
P(\hat{X} \neq p_1 \mid X = p_1) &= P(\{N_R > \tfrac{d}{2}\} \cup \{N_I < -\tfrac{d}{2}\}) \\
&= P\left(\left\{\frac{N_R}{\sqrt{N_0/2}} > \frac{d/2}{\sqrt{N_0/2}}\right\} \cup \left\{\frac{N_I}{\sqrt{N_0/2}} < -\frac{d/2}{\sqrt{N_0/2}}\right\}\right) \\
&\leq P\left(\frac{N_R}{\sqrt{N_0/2}} > \frac{d/2}{\sqrt{N_0/2}}\right) + P\left(\frac{N_I}{\sqrt{N_0/2}} < -\frac{d/2}{\sqrt{N_0/2}}\right) \\
&= 2\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right).
\end{aligned} \tag{1}$$

When $X$ is one of the two interior points, say $p_2$, the prob. of error is

$$\begin{aligned}
&P(\hat{X} \neq p_2 \mid X = p_2) = P(\{N_R > \tfrac{d}{2}\} \cup \{N_R < \tfrac{-d}{2}\} \cup \{N_I < \tfrac{-d}{2}\}) \\
&\leq P\left(\frac{N_R}{\sqrt{N_0/2}} > \frac{d/2}{\sqrt{N_0/2}}\right) + P\left(\frac{N_R}{\sqrt{N_0/2}} < \frac{-d/2}{\sqrt{N_0/2}}\right) + P\left(\frac{N_I}{\sqrt{N_0/2}} < -\frac{d/2}{\sqrt{N_0/2}}\right) \\
&= 3\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right).
\end{aligned} \tag{2}$$

The average probability of error is

$$\begin{aligned}
P_e &= \frac{1}{6}\left(4 \cdot 2\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right) + 2 \cdot 3\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)\right) \\
&= \frac{7}{3}\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right) = \frac{7}{3}\mathcal{Q}\left(\sqrt{\frac{6\log_2 6}{11}\frac{E_b}{N_0}}\right),
\end{aligned}$$

where the last equality is obtained using part (a).(i).

iv) Since the points on a PSK constellation are located on a circle, they have equal energy, equal to the square of the radius. In this case, the energy is $11d^2/12$, so radius is $\sqrt{11/12}\,d$. Since adjacent points are 60° apart, the distance between adjacent points is $\sqrt{11/12}\,d$. [10%]

v) The argument of the $Q$-function, which determines the error probability, depends on the distance between adjacent points in the constellation. For the 6-PSK, the distance between adjacent points is smaller than for 6-QAM, i.e., $\sqrt{11/12}\,d$ vs $d$.

Each point in the 6-PSK constellation has two nearest neighbours (at distance $\sqrt{11/12}\,d$). For 6-QAM four points have two nearest neighbours and the other two have three nearest neighbours, at distance $d$. Therefore, for 6-QAM, the constant multiplying the $Q$ function will be larger and the argument of the Q-function is smaller. Since $Q(x)$ decays exponentially with $x^2$ (recall that $Q(x) \leq e^{-x^2/2}$ for $x > 0$), 6-QAM will have smaller error probability (except possibly for very small values of $E_b/N_0$ when the exponential is very close to 1 in both cases). [15%]

(b) The optimal detection rule is the MAP rule: [20%]

$$\begin{aligned}\hat{X} &= \arg\max_{x\in\{3,-1\}} \; P(Y=y|X=x)\; P(X=x) \\ &= \arg\max_{x\in\{3,-1\}} \; e^{-(y-x)^2/(2\sigma^2)} P(X=x)\end{aligned} \tag{3}$$

For $x = 3$, the test statistic in (3) is $pe^{-(y-3)^2/(2\sigma^2)}$. For $x=-1$, the test statistic is $(1-p)e^{-(y+1)^2/(2\sigma^2)}$. Therefore, $\hat{X} = 3$ when

$$\begin{aligned} pe^{-(y-3)^2/(2\sigma^2)} \geq (1-p)e^{-(y+1)^2/(2\sigma^2)} &\Leftrightarrow \ln p - \frac{(y-3)^2}{2\sigma^2} \geq \ln(1-p) - \frac{(y+1)^2}{2\sigma^2} \\ &\Leftrightarrow \ln p + \frac{6y-9}{2\sigma^2} \geq \ln(1-p) - \frac{(2y+1)}{2\sigma^2} \\ &\Leftrightarrow y \geq 1 + \frac{2\sigma^2}{8} \ln \frac{(1-p)}{p}. \end{aligned} \tag{4}$$

Therefore the optimal decision rule is

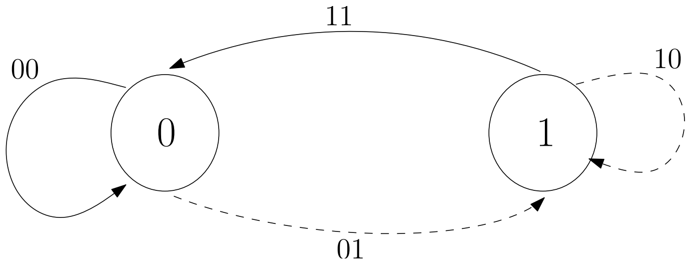
$$\hat{X} = \begin{cases} A & \text{if } Y \geq T \\ -A & \text{if } Y < T \end{cases}$$

where $T = 1 + \frac{\sigma^2}{4} \ln \frac{(1-p)}{p}$.

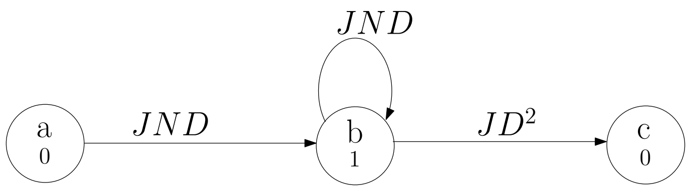
***Assessor's comment:*** *Generally well answered. For part (a).(v), some did not recognise that the argument of the Q-function is the key to determining the error probability rather than the average number of nearest neighbours.*

## Question 3

(a) The state diagram of the code is shown below. Transitions due to input bit 0 are shown in solid lines, and those due to input 1 are shown in dashed lines. The edges are labeled with the code bits corresponding to the transitions. [15%]




(b) The state diagram with the the edges labeled with the appropriate powers of $J$, $N$, $D$ is shown below. [15%]




We have

$$X_b = JND(X_a + X_b)$$
$$X_c = JD^2 X_b$$

Eliminating $X_b$, we obtain the transfer function between $X_a$ and $X_c$ as

$$\frac{X_c}{X_a} = \frac{J^2ND^3}{1 - JND}$$
$$= J^2ND^3 + J^3N^2D^4 + J^4N^2D^5 + \dots$$

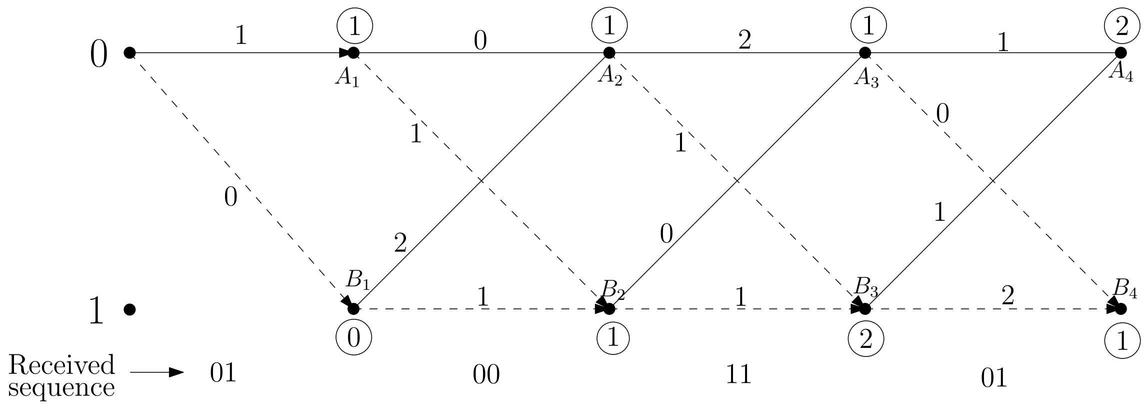
(c) Each term in the expansion of the transfer function represents a path that starts and ends at the zero state, with at least one non-zero state in between. The exponent of $J$ specifies the number of branches in the path before it returns to the zero state; the exponent of $D$ in each term specifies the Hamming distance between the output along that path and the all-zero codeword; the exponent of $N$ is the total number of ones in the input along that path.

For the above code, the first term in the transfer function expansion tells us that there is one path with two steps before it returns to the all-zero state, with a single 1 input and three 1 outputs along the path (path *abc*). The second term tells us that there is one path with three branches before returning to the all-zero state, with two 1 inputs and four 1 outputs along that path (path *abbc*). [20%]

(d) The free distance of a convolutional code is the minimum Hamming distance between any two code-sequences of the convolutional code. It is given by the minimum $D$ exponent in the transfer function. The free distance of the above code is therefore 3. [5%]

(e) We use the Viterbi algorithm. The trellis representation of the code is shown in the Figure below. [30%]

- The numbers on the edges indicate the distance of the output of the transition from the corresponding bits of the received sequence (which are shown at the bottom of the trellis).




- The numbers that are encircled at the nodes indicate the *minimum distance* of the corresponding node from the origin (node 0).

We see that the minimum distance path is $0 - B_1 - B_2 - A_3 - B_4$.

The decoded sequence is 01 10 11 01.

The corresponding input sequence is 1 1 0 1.

(f) A convolutional encoder is catastrophic if there exists two (arbitrarily long) code sequences which differ only in a small number of bits, but the corresponding two input sequences differ in an arbitrarily large number of bits. With a catastrophic encoder, when one of these code sequences is transmitted a small number of channel errors can lead to an arbitrarily large number of errors in the input sequence.

(An excellent answer might give an example, like the one in the lecture notes.) [15%]

***Assessor's comment:*** *Generally well answered. Some had trouble recalling what exactly a catastrophic encoder meant, and just said "a code where a finite number of channel errors can cause an infinite number of decoding errors". Many gave the condition (in terms of the generator polynomials) for the encoder to not be catastrophic.*

## Question 4

(a) i) The bandwidth of the OFDM signal is $W = \frac{N}{T}$, where $N$ is the number of sub-carriers, and $\frac{1}{T}$ is the spacing between sub-carriers. We are given that $N = 128$, and $\frac{1}{T} = 10^5$ Hz. [10%]

Therefore $W = 12.8$ MHz.

ii) If the filter has coefficients $h[0], \dots, h[L]$, then the output at time $k$ is

$$y[k] = h[0]x[k] + \sum_{\ell=1}^{L} h[k]x[k-\ell]. \quad (5)$$

The duration of the guard interval is [10%]

$$\Delta = \frac{LT}{N} = \frac{40 \times 10^{-5}}{128} = 3.125 \times 10^{-6} \text{ sec.}$$

iii) In a time interval $(T + \Delta)$, the OFDM signal carries $N$ 16-QAM symbols, Therefore, the coded bit rate of the OFDM signal is [25%]

$$\frac{N \times 4}{T + \Delta} \text{ symbols /sec} = \frac{128 \times 4}{13.125 \times 10^{-6}} \text{ bits/sec} = 39.0095 \text{ Mbits/sec.}$$

Since we use a code of rate 3/4, the user data rate is

$$\frac{3}{4} \times 39.0095 \text{ Mbits/sec} = 29.2571 \text{ Mbits/sec.}$$

iii) The role of the cyclic prefix is to eliminate inter-symbol interference between adjacent sub-carriers. To avoid ISI, we want the OFDM information sequence in the frequency domain to be each multiplied by a single DFT coefficient of the channel filter. Equivalently, the time-domain version of the information sequence should undergo circular convolution with the channel filter. Since the channel actually acts on the input via linear convolution in the time domain (as per (5)), the cyclic prefix mimics circular convolution by transmitting the appropriate symbols in a guard interval before the block of information symbols is transmitted in the OFDM symbol period. [25%]

In particular, just before transmitting the block of time-domain symbols $x[0], \dots, x[N-1]$ over a duration of length $T$, we transmit $x[-L], \dots, x[-1]$ in a guard interval of length $LT/N$, where

$$x[-L] = x[N-L], \; \dots, \; x[-1] = x[N-1].$$

(b) The steps in Dijkstra's algorithm are illustrated in the table below. In each iteration of the algorithm, one node is added to the set $\mathcal{K}$ and the variables $\omega_{ai}, p_{a_i}$ are updated. $\omega_{ai}$ is the current minimum cost from node $a$ to $i$; $p_{ai}$ is the previous hop to node $i$ in the min-cost path from $a$.

The table is [40%]

| $\mathcal{K}$ | $(\omega_{ab}, p_{ab})$ | $(\omega_{ac}, p_{ac})$ | $(\omega_{ad}, p_{ad})$ | $(\omega_{ae}, p_{ae})$ | $(\omega_{af}, p_{af})$ |
|---|---|---|---|---|---|
| $\{a\}$ | $(3, a)$ | $(5, a)$ | $(\infty, -1)$ | $(\infty, -1)$ | $(\infty, -1)$ |
| $\{a, b\}$ | · | $(5, a)$ | $(7, b)$ | $(5, b)$ | $(\infty, -1)$ |
| $\{a, b, c\}$ | · | · | $(7, b)$ | $(5, b)$ | $(\infty, -1)$ |
| $\{a, b, c, e\}$ | · | · | $(6, e)$ | · | $(8, e)$ |
| $\{a, b, c, e, d\}$ | · | · | · | · | $(8, e)$ |
| $\{a, b, c, e, d, f\}$ | · | · | · | · | · |

From the table, we find that the minimum cost paths from node $a$ are:

a - b (cost 3)

a - c (cost 5)

a - b - e - d (cost 6)

a - b - e (cost 5)

a - b - e - f (cost 8)

***Assessor's comment:*** *Generally well answered, though many had trouble calculating the duration of the guard interval.*