Version MAG/3

EGT2
ENGINEERING TRIPOS PART IIA

Monday 29 April 2024 14.00 to 15.40

Module 3M1

MATHEMATICAL METHODS

Answer all the questions.

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed.
Engineering Data Books.

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 For $x \in \mathbb{R}$ with probability density function $p(x)$ and function $f(x)$ the integral

$$\mathcal{I} = \int_{-\infty}^{\infty} f(x)p(x)dx$$

can be approximated by a general Monte Carlo estimate

$$\hat{\mathcal{I}} = \sum_{n=1}^{N} w_n f(x_n)$$

where each x_n is distributed as $p(x)$, noting that these samples may be correlated. This can be written in the vector form

$$\hat{\mathcal{I}} = \mathbf{w}^{\mathsf{T}}\mathbf{f}$$

by defining the N-dimensional column vectors $\mathbf{f} = [f(x_1), f(x_2), \cdots, f(x_N)]^{\mathsf{T}} \in \mathbb{R}^N$ and $\mathbf{w} = [w_1, w_2, \cdots, w_N]^{\mathsf{T}}$ such that $\sum_{n=1}^{N} w_n = \mathbf{1}^{\mathsf{T}}\mathbf{w} = 1$, where the N-dimensional vector of ones is $\mathbf{1} = [1, 1, \cdots, 1]^{\mathsf{T}}$.

(a) Noting that the variance of $\hat{\mathcal{I}}$ can be written as $\mathbf{w}^{\mathsf{T}}\Sigma\mathbf{w}$ where Σ is the covariance matrix of the vector $\mathbf{f}$, derive the Lagrangian, $\mathcal{L}(\mathbf{w}, \lambda)$, corresponding to the constrained optimisation problem to find the $\mathbf{w}$ that minimises the variance of $\hat{\mathcal{I}}$. [10%]

(b) Denoting the N-dimensional column vector of zeros as $\mathbf{0}$, rewrite the two systems of equations

$$\begin{aligned} \frac{\partial}{\partial \mathbf{w}}\mathcal{L}(\mathbf{w}, \lambda) &= \mathbf{0} \\ \mathbf{1}^{\mathsf{T}}\mathbf{w} &= 1 \end{aligned}$$

in matrix form, $\mathbf{A}\mathbf{c} = \mathbf{d}$, explicitly stating the form of matrix $\mathbf{A}$ when the two $N+1$ column vectors are given as $\mathbf{c} = [\mathbf{w}^{\mathsf{T}}, \lambda]^{\mathsf{T}}$ and $\mathbf{d} = [\mathbf{0}^{\mathsf{T}}, 1]^{\mathsf{T}}$. [20%]

(c) Noting that the matrix $\mathbf{A}$ and its inverse can be written in partitioned form such that

$$\mathbf{A}^{-1} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{C}_1^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{C}_2^{-1} \\ -\mathbf{C}_2^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{C}_2^{-1} \end{pmatrix}$$

where $\mathbf{C}_1 = \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}$ and $\mathbf{C}_2 = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$, solve the system for $\mathbf{c} = [\mathbf{w}^{\mathsf{T}}, \lambda]^{\mathsf{T}}$ and state the conditions for which this solution is unique. [40%]

(d) Show that the variance minimising Monte Carlo estimator $\hat{\mathcal{I}}^*$ for $\mathcal{I}$ takes the form

$$\hat{\mathcal{I}}^* = \frac{\mathbf{1}^\top \Sigma^{-1} \mathbf{f}}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}}$$

and the corresponding minimal variance is equal to

$$\frac{1}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}}$$

[20%]

(e) If the x_n are sampled independently and identically from $p(x)$ the covariance matrix for $\mathbf{f}$ takes the form $\sigma_f^2 \mathbf{I}$, where $\mathbf{I}$ is an $N \times N$ dimensional identity matrix, show that the minimum variance estimator is of the form $\hat{\mathcal{I}}^* = \frac{1}{N} \sum_{n=1}^{N} f(x_n)$. [10%]

2 The singular value decomposition of a matrix $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ is $\boldsymbol{U\Sigma V}^H$, where the columns of $\boldsymbol{U}$ are the 'left singular vectors' and the columns of $\boldsymbol{V}$ are the 'right singular vectors'.

(a) In $\mathbb{R}^2$, sketch the unit sphere S in the l_2-norm. [10%]

(b) For a matrix $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ with non-zero singular values $\sigma_1 > \sigma_2$ and singular vectors that are not aligned with the coordinate axes, sketch the result of applying $\boldsymbol{A}$ to the unit sphere S. Annotate your sketch to give a geometric interpretation of the singular values and the left and right singular vectors. [20%]

(c) Give expressions for the determinant and the condition number κ_2 for a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ in terms of the singular values of $\boldsymbol{A}$. For $m = 2$ and with reference to the geometry of the unit sphere S after $\boldsymbol{A}$ is applied to it, give geometric interpretations (in words) of the determinant and the condition number. [20%]

(d) An approximation to a matrix $\boldsymbol{A} \in \mathbb{C}^{m\times n}$, $m \geq n$, is given by

$$\boldsymbol{A}_k = \sum_{i=1}^{k} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^H,$$

where $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ are left and right singular vectors, respectively, and $k < n$.

(i) What is the rank of $\boldsymbol{A}_k$? [10%]

(ii) Recalling the definition of the Frobenius norm, $\|\boldsymbol{A}\|_F^2 := \sum_{i,j} |A_{ij}|^2 = \mathrm{trace}(\boldsymbol{A}\boldsymbol{A}^H)$, give an expression for $\|\boldsymbol{A} - \boldsymbol{A}_k\|_F$ in terms of the singular values of $\boldsymbol{A}$. [20%]

(iii) Given the result $\sigma_i(\boldsymbol{X}) + \sigma_j(\boldsymbol{Y}) \geq \sigma_{i+j-1}(\boldsymbol{X} + \boldsymbol{Y})$, $i, j \geq 1$, where $\sigma_i(\boldsymbol{X})$ is the ith singular value of $\boldsymbol{X}$, prove that $\boldsymbol{A}_k$ is the best rank k approximation of $\boldsymbol{A}$ in the Frobenius norm. [20%]

3 The behaviour of bacteria is described by a birth-death process. For an individual bacterium λ is the *birth rate* and μ the *death rate* per unit time. Initially the process starts with a single bacterium at $t = 0$.

(a) Draw the state-space associated with the number of bacteria up to 4 and the transition rates between these states. [10%]

(b) The change in the probability distribution of the number of bacteria can be described by the following equation

$$\frac{d\boldsymbol{\pi}(t)}{dt} = \boldsymbol{\pi}(t)\mathbf{Q}$$

(i) Show that the following expression is satisfied for the change in the probability of there being two bacteria in time Δt

$$\pi_2(t + \Delta t) \approx (1 - 2\lambda\Delta t - 2\mu\Delta t)\pi_2(t) + \lambda\Delta t\pi_1(t) + 3\mu\Delta t\pi_3(t)$$

where $\pi_i(t)$ is the i-th element of row vector $\boldsymbol{\pi}(t)$. You should clearly state any assumptions that are made in this derivation. [15%]

(ii) Hence find the elements of the first four rows of the transition rate matrix $\mathbf{Q}$. [15%]

(c) For this process the distribution of the number of bacteria at time t is given by

$$\pi_0(t) = \alpha(t), \quad \pi_i(t) = (1 - \alpha(t))\,(1 - \beta(t))\,\beta(t)^{i-1} \text{ for } i > 0$$

where

$$\alpha(t) = \frac{\mu\,(\exp((\lambda - \mu)t) - 1)}{\lambda\exp((\lambda - \mu)t) - \mu}, \quad \beta(t) = \frac{\lambda}{\mu}\alpha(t)$$

(i) Show that this expression is consistent with the initial conditions at $t = 0$. Note the equality $0^0 = 1$. [10%]

(ii) Derive the condition in terms of μ and λ that must be satisfied to guarantee that the bacteria will eventually die out. [25%]

(iii) Derive an expression for the probability that the bacteria will eventually die out. This expression should be given in terms of μ and λ and simplified where possible. [10%]

(iv) If the initial number of bacteria at $t = 0$ is N how would this alter the probability that the bacteria will eventually die out. [15%]

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