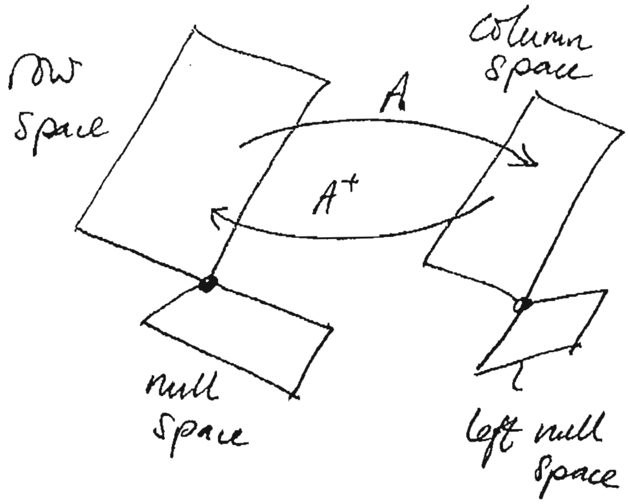


1.

(a) If $A = \hat{Q}\Sigma Q^t$ is a singular value decomposition of A, then $A^+ = Q\Sigma^+\hat{Q}^t$

where $\Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_1} & & & & \\ & \frac{1}{\sigma_2} & & 0 & \\ & & \ddots & & \\ & & & \frac{1}{\sigma_r} & \\ & 0 & & & 0 \\ & & & & 0\;0\;0 \end{bmatrix}$

and σ_i = singular values of A, r = rank of A

A^+ is the inverse mapping of that associated with A mapping row space to column space.

If $A\underline{x} = \underline{b}$ is to be solved in a least-squares sense, $\underline{x} = A^+\underline{b}$ is the least-squares solution of minimal length.

In outer product notation:

$$A = \sigma_1 \hat{\underline{q}}_1 \underline{q}_1^t + \sigma_2 \hat{\underline{q}}_2 \underline{q}_2^t + \dots + \sigma_r \hat{\underline{q}}_r \underline{q}_r^t$$

$$A^+ = \frac{1}{\sigma_1} \underline{q}_1 \hat{\underline{q}}_1^t + \frac{1}{\sigma_2} \underline{q}_2 \hat{\underline{q}}_2^t + \dots + \frac{1}{\sigma_r} \underline{q}_r \hat{\underline{q}}_r^t$$ [15%]

(b) Performing svd of A

$$A^TA = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & -1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 5/2 \end{bmatrix}$$

Already diagonal $\Rightarrow$ $\sigma_1^2 = 7$ $\sigma_2^2 = 5/2$

$\underline{q}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\underline{q}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\hat{\underline{q}}_1 = \frac{A\underline{q}_1}{\sigma_1} = \frac{1}{\sqrt{7}}\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{7}}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{\underline{q}}_2 = \frac{A\underline{q}_2}{\sigma_2} = \sqrt{\frac{2}{5}}\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \sqrt{\frac{2}{5}}\begin{bmatrix} 1 \\ -1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\therefore A^+ = \frac{1}{\sqrt{7}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\frac{1}{\sqrt{7}}\begin{bmatrix} 2 & 1 & 1 & 1 \end{bmatrix} + \sqrt{\frac{2}{5}}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{2}{\sqrt{5}}\begin{bmatrix} 1 & -1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{2}{5} & \frac{-2}{5} & \frac{-1}{5} & \frac{-1}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\ \frac{2}{5} & \frac{-2}{5} & \frac{-1}{5} & \frac{-1}{5} \end{bmatrix}$$

Least squares solution is, therefore,

$$\underline{x} = A^+\underline{b} = \begin{bmatrix} \frac{2}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\ \frac{2}{5} & \frac{-2}{5} & \frac{-1}{5} & \frac{-1}{5} \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4+3-1+1}{7} \\ \frac{4-6+1-1}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -\frac{2}{5} \end{bmatrix}$$

[50%]

(c) l_2-norm of a matrix = principal singular value

S. values of $A^+ = \dfrac{1}{\text{s.values of } A} = \dfrac{1}{\sqrt{7}}$ & $\sqrt{\dfrac{2}{5}}$

$\therefore \quad \|A^+\| = \sqrt{\dfrac{2}{5}} = \dfrac{1}{\sigma_2}$ [15%]

(d) $A^+ = \dfrac{1}{\sigma_1} \underline{q}_1 \underline{\hat{q}}_1^t + \dfrac{1}{\sigma_2} \underline{q}_2 \underline{\hat{q}}_2^t$ where σ_i = s.values of A

Seek $\underline{x}$ s.t. $\|A^+\underline{x}\| = \dfrac{1}{\sigma_2}\|\underline{x}\|$

This will be true for $\underline{x} = \underline{\hat{q}}_2$ (or any multiple of it)

$A^+\underline{\hat{q}}_2 = \dfrac{1}{\sigma_2}\underline{q}_2 \Rightarrow \|A^+\underline{\hat{q}}_2\| = \dfrac{1}{\sigma_2}\|\underline{q}_2\|$

& $\underline{q}_2$ is a unit vector (as is $\underline{\hat{q}}_2$).

$\therefore \quad \|A^+\underline{\hat{q}}_2\| = \|A^+\|\,\|\underline{\hat{q}}_2\|$

[20%]

Q2

(a) For a minimum, the gradient of U is zero and its Hessian is positive definite.

$$\frac{\partial U}{\partial x_1} = k_2 x_1 - k_3(x_2 - x_1) = 0$$

$$\therefore \quad (k_2 + k_3)x_1 = k_3 x_2 \quad \Rightarrow \quad x_1 = \frac{k_3 x_2}{k_2 + k_3}$$

$$\frac{\partial U}{\partial x_2} = k_3(x_2 - x_1) + k_1 x_2 - P = 0$$

$$\therefore \quad (k_3 + k_1)x_2 - k_3 x_1 = P$$

$$\therefore \quad (k_3 + k_1)x_2 - k_3 \frac{k_3 x_2}{k_2 + k_3} = P$$

$$\therefore \quad (k_2 + k_3)(k_3 + k_1)x_2 - k_3^2 x_2 = P(k_2 + k_3)$$

$$\therefore \quad (k_1 k_2 + k_1 k_3 + k_2 k_3)x_2 = P(k_2 + k_3)$$

$$\therefore \quad x_2^* = \frac{P(k_2 + k_3)}{k_1 k_2 + k_1 k_3 + k_2 k_3} \quad \Rightarrow \quad x_1^* = \frac{k_3 x_2^*}{k_2 + k_3} = \frac{P k_3}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

Need to check that the Hessian is positive definite.

$$\frac{\partial^2 U}{\partial x_1^2} = k_2 + k_3; \quad \frac{\partial^2 U}{\partial x_1 \partial x_2} = -k_3; \quad \frac{\partial^2 U}{\partial x_2^2} = k_1 + k_3$$

Hence
$$H = \begin{bmatrix} k_2 + k_3 & -k_3 \\ -k_3 & k_1 + k_3 \end{bmatrix}$$

The determinants of the principal minor matrices are

$$J_1 = k_2 + k_3$$

and
$$J_2 = (k_2 + k_3)(k_1 + k_3) - k_3^2 = k_1 k_2 + k_1 k_3 + k_2 k_3$$

Both principal minor matrices are clearly positive for positive k_is, so H is positive definite and this is indeed a minimum. [30%]

(b) Newton's method gives $\quad \mathbf{x}_{i+1} = \mathbf{x}_i - H(\mathbf{x}_i)^{-1} \nabla U(\mathbf{x}_i)$

Using the results from (a) $\quad \nabla U = \begin{bmatrix} (k_2 + k_3)x_1 - k_3 x_2 \\ -k_3 x_1 + (k_1 + k_3)x_2 - P \end{bmatrix}$

$$H = \begin{bmatrix} k_2 + k_3 & -k_3 \\ -k_3 & k_1 + k_3 \end{bmatrix} \quad \Rightarrow \quad H^{-1} = \frac{1}{k_1 k_2 + k_1 k_3 + k_2 k_3} \begin{bmatrix} k_1 + k_3 & k_3 \\ k_3 & k_2 + k_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\text{new}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \frac{1}{k_1 k_2 + k_1 k_3 + k_2 k_3} \begin{bmatrix} k_1 + k_3 & k_3 \\ k_3 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} (k_2 + k_3)x_1 - k_3 x_2 \\ -k_3 x_1 + (k_1 + k_3)x_2 - P \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{new} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} -$$

$$\frac{1}{k_1k_2 + k_1k_3 + k_2k_3} \begin{bmatrix} (k_1 + k_3)(k_2 + k_3)x_1 - k_3(k_1 + k_3)x_2 - k_3^2 x_1 + k_3(k_1 + k_3)x_2 - k_3 P \\ k_3(k_2 + k_3)x_1 - k_3^2 x_2 - k_3(k_2 + k_3)x_1 + (k_1 + k_3)(k_2 + k_3)x_2 - (k_2 + k_3)P \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{new} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \frac{1}{k_1k_2 + k_1k_3 + k_2k_3} \begin{bmatrix} (k_1k_2 + k_1k_3 + k_2k_3)x_1 - k_3 P \\ (k_1k_2 + k_1k_3 + k_2k_3)x_2 - (k_2 + k_3)P \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{new} = \frac{1}{k_1k_2 + k_1k_3 + k_2k_3} \begin{bmatrix} (k_1k_2 + k_1k_3 + k_2k_3)x_1 - (k_1k_2 + k_1k_3 + k_2k_3)x_1 + k_3 P \\ (k_1k_2 + k_1k_3 + k_2k_3)x_2 - (k_1k_2 + k_1k_3 + k_2k_3)x_2 + (k_2 + k_3)P \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{new} = \frac{1}{k_1k_2 + k_1k_3 + k_2k_3} \begin{bmatrix} k_3 P \\ (k_2 + k_3)P \end{bmatrix}$$

This is as expected as the problem is quadratic. Newton's Method converges in one iteration on quadratic problems. [30%]

(c) For the values given

$$\nabla U = \begin{bmatrix} 5x_1 - 3x_2 \\ -3x_1 + 4x_2 - 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 5 & -3 \\ -3 & 4 \end{bmatrix}$$

From the initial solution $(x_1, x_2) = (0,0)$

$$\mathbf{d}_0 = -\nabla U(\mathbf{x}_0) = -\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\alpha_0 = \frac{\mathbf{d}_0^T \mathbf{d}_0}{\mathbf{d}_0^T H \mathbf{d}_0} = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \frac{1}{\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 4 \end{bmatrix}} = \frac{1}{4}$$

$$\mathbf{x}_1 = \mathbf{x}_0 + \alpha_0 \mathbf{d}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.25 \end{bmatrix}$$

For the second iteration

$$\nabla U(\mathbf{x}_1) = \begin{bmatrix} -3 \times 0.25 \\ 4 \times 0.25 - 1 \end{bmatrix} = \begin{bmatrix} -0.75 \\ 0 \end{bmatrix}$$

$$\beta_0 = \left[\frac{|\nabla U(\mathbf{x}_1)|}{|\nabla U(\mathbf{x}_0)|}\right]^2 = \left[\frac{0.75}{1}\right]^2 = 0.5625$$

$$\mathbf{d}_1 = -\nabla U(\mathbf{x}_1) + \beta_0 \mathbf{d}_0 = -\begin{bmatrix} -0.75 \\ 0 \end{bmatrix} + 0.5625\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.5625 \end{bmatrix}$$

$$\alpha_1 = -\frac{\mathbf{d}_1^T \nabla U(\mathbf{x}_1)}{\mathbf{d}_1^T H \mathbf{d}_1} = -\frac{\begin{bmatrix}0.75 & 0.5625\end{bmatrix}\begin{bmatrix}-0.75\\0\end{bmatrix}}{\begin{bmatrix}0.75 & 0.5625\end{bmatrix}\begin{bmatrix}5 & -3\\-3 & 4\end{bmatrix}\begin{bmatrix}0.75\\0.5625\end{bmatrix}} = \frac{0.5625}{\begin{bmatrix}0.75 & 0.5625\end{bmatrix}\begin{bmatrix}2.0625\\0\end{bmatrix}} = \frac{4}{11}$$

$$\mathbf{x}_2 = \mathbf{x}_1 + \alpha_1 \mathbf{d}_1 = \begin{bmatrix}0\\0.25\end{bmatrix} + \frac{4}{11}\begin{bmatrix}0.75\\0.5625\end{bmatrix} = \begin{bmatrix}0.272727\\0.454545\end{bmatrix} \text{ or } \begin{bmatrix}\frac{3}{11}\\\frac{5}{11}\end{bmatrix}$$

This is the location of the potential energy minimum (easily shown by substituting values in the expressions obtained in (a) or (b)).

The behaviour is as expected. On quadratic problems the CGM converges in a number of iterations equal to the number of control variables (two here). [40%]

Q3

(a) Using the minimum matrix method, at each step as much flow as possible is allocated to the available arc with the lowest cost. The tableau below summarises the steps of the minimum matrix method in this case.

Factories	Distribution centres D	E	F	Supply
A	50	180	160	~~200~~ 50
	150 [step 1]	50 [step 4]		
B	80	300	60	~~200~~
			200 [step 2]	
C	125	310	100	~~300~~ 200
		200 [step 5]	100 [step 3]	
Demand	~~150~~	~~250~~ ~~200~~	~~300~~ ~~100~~	

This matches the solution shown on the network diagram in the question. [20%]

(b) The first step is to calculate the reduced costs using the standard formulae:

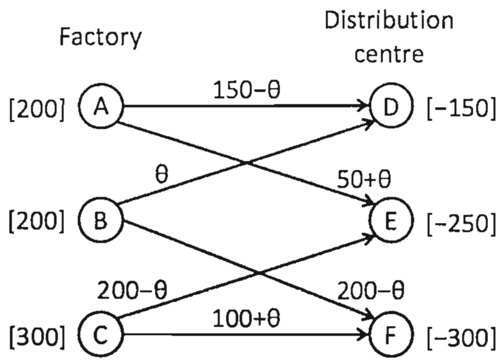
$u_i + v_j = c_{ij}$ to find the simplex multipliers associated with the current basis, and

$\bar{c}_{ij} = c_{ij} - u_i - v_j$ to find the reduced costs for all non-basic variables.

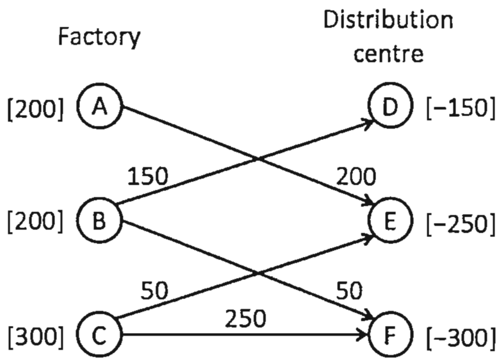
Doing this on the tableau associated with the initial feasible basis given by the minimum matrix method yields:

Factories	Distribution centres D	E	F	u_i
A	50	180	160	0
	—	—	190	
B	80	300	60	90
	–60	30	—	
C	125	310	100	130
	–55	—	—	
v_j	50	180	–30	

The lowest reduced cost is for arc BD. Introducing this into the basis:



The largest possible value of θ is 150, giving:



Repeating the process of finding the reduced costs for the new basis:

Factories	Distribution centres			u_i
	D	E	F	
A	50	180	160	0
	60	—	190	
B	80	300	60	90
	—	30	—	
C	125	310	100	130
	5	—	—	
v_j	−10	180	−30	

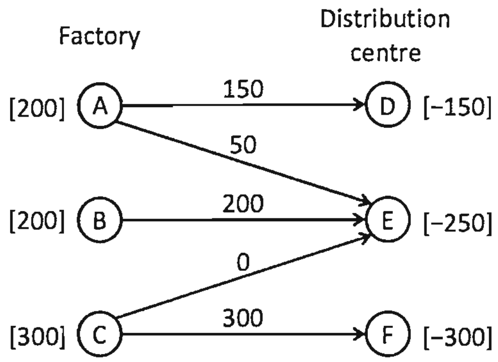
As there are no negative reduced costs, this is the minimum. [50%]

(c) The *northwest-corner method* ignores costs entirely. It starts with the upper leftmost corner (the northwest corner) and assigns the maximum possible flow allocation to that cell. Then it moves to the right, if there is any remaining supply in the first row, or to the next lower cell, if there is any remaining demand in the first column, and assigns the maximum possible flow allocation to that cell. The procedure repeats itself until the lowest right corner is reached, at which point all the supply is exhausted and all the demand satisfied.

The table overleaf summarises the steps of the northwest-corner method, in terms of the transportation tableau, in this case.

Factories	Distribution centres			Supply
	D	E	F	
A	150 [step 1]	50 [step 2]		~~200~~ ~~50~~
B		200 [step 3]		~~200~~
C			300 [step 4]	~~300~~
Demand	~~150~~	~~250~~ ~~200~~	~~300~~	

However, this alone does not constitute a spanning tree – the nodes associated with factory C and distribution centre F are disconnected from the other nodes. To proceed to solve the problem using the standard method (not required), an arc with zero flow must be added between node C and node D or node E or between node F and node A or node B, e.g. as shown below.



The minimum matrix method is generally preferred as a method for generating an initial feasible basis because, unlike the northwest-corner method, it takes costs into consideration. It is therefore likely that the initial feasible basis the minimum matrix method yields will be closer to the optimum, and fewer iterations of the simplex method will be needed to solve the problem at hand. [30%]

4. (a) The elements of the j'th row represent the probability of moving to 1, 2, ..., N at the next step. Since we have to move somewhere & these exhaust the possibilities,

$$\sum \text{probabilities} = 1 = \sum_{k=1}^{N} p_{jk}$$

[10%]

(b) $$P - I = \begin{bmatrix} p_{11}-1 & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22}-1 & \cdots & p_{2N} \\ \vdots & & & \\ p_{N1} & p_{N2} & \cdots & p_{NN}-1 \end{bmatrix}$$

The sum of the first N-1 columns $\begin{cases} = \sum_{k=1}^{N-1} p_{jk} - 1 & \text{for } j \le N-1 \\ = \sum_{k=1}^{N-1} p_{jk} & \text{for } j = N \end{cases}$

This is − last column.

i.e. columns of $P-I$ not independent

$\Rightarrow$ $P-I$ is singular, $\det(P-I) = 0$ i.e. $\lambda = 1$ is an e-value

[20%]

(c) Let q_i = expected no of transitions to get to 1, given we start at i

$$\therefore \quad q_1 = a.1 + (1-a)\, q_2 \qquad (a)$$

$$q_2 = \tfrac{1}{2}(1+q_2) + \tfrac{1}{2}(1+q_3) \qquad (b)$$

$$q_3 = \tfrac{1}{2}(1+q_3) + \tfrac{1}{2} \qquad (c)$$

(c) $\Rightarrow \frac{1}{2} q_3 = 1 \quad \Rightarrow \quad q_3 = 2$

(b) $\Rightarrow \frac{1}{2} q_2 = 1 + \frac{1}{2} q_3 \Rightarrow q_2 = 4$

(a) $\Rightarrow q_1 = a + (1-a)4 = \underline{\underline{4 - 3a}}$

[40%]

(d) q_1 now altered to preclude staying at 1 as a return to 1. Once having moved away from 1, must go through 3 to return to 1.

q_2 & q_3 are unchanged since $2 \to 1$ involves going through 3.

$$\therefore \quad q_1^{\text{via } 3} = a\left(1 + q_1^{\text{via } 3}\right) + (1-a) q_2$$

$$\Rightarrow q_1^{\text{via } 3}(1-a) = a + (1-a)4$$

$$\Rightarrow q_1^{\text{via } 3} = \frac{4 - 3a}{1-a}$$

[15%]

(e) At equilibrium, distribution is e-vector corresponding to e-value 1

$$\therefore \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & 1-a & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} ax + \tfrac{1}{2} z &= x \\ (1-a)x + \tfrac{y}{2} &= y \\ \tfrac{y}{2} + \tfrac{z}{2} &= z \end{aligned}\right\} \Rightarrow y = z = 2(1-a)x$$

& $x + y + z = 1$

$$\Rightarrow x = \frac{1}{5-4a} \qquad y = \frac{2(1-a)}{5-4a} = z$$

[15%]