3M1 2021 Q1

a) i) $\|\underline{x}\|_1 = 1 + |1-i| + |1+i|$
$= 1 + \sqrt{2} + \sqrt{2} = \underline{1 + 2\sqrt{2}}$

$\|\underline{x}\|_2 = (1 + 2 + 2)^{\frac{1}{2}} = \sqrt{5}$

$\|\underline{x}\|_\infty = \sqrt{2}$

ii) $\|\underline{x}\|_1 = \sum_{r=1}^{n} \left|\sqrt{(-1)^r}\right| = n$

$\|\underline{x}\|_2 = \left(\sum_{r=1}^{n} \left|\sqrt{(-1)^r}\right|^2\right)^{\frac{1}{2}} = \left(\sum_{r=1}^{n} 1\right)^{\frac{1}{2}} = \sqrt{n}$

$\|\underline{x}\|_\infty = 1$

b) i) $[\underline{\underline{A}}\,\underline{\underline{A}}^H]_{ij} = A_{ik}\bar{A}_{jk}$

$\underbrace{\sum_i [\underline{\underline{A}}\,\underline{\underline{A}}^H]_{ii}}_{Tr(\underline{\underline{A}}\,\underline{\underline{A}}^H)} = \sum_i \sum_k A_{ik}\bar{A}_{ik} = \|\underline{\underline{A}}\|_F^2$

* Note that $\|\underline{\underline{A}}\|_F = \|\underline{\underline{A}}^H\|_F$
$\Rightarrow Tr(\underline{\underline{A}}\,\underline{\underline{A}}^H) = Tr(\underline{\underline{A}}^H\underline{\underline{A}})$ (1)

ii) $\|\underline{\underline{A}}\,\underline{\underline{U}}\|_F^2 = \mathrm{Tr}\left(\underline{\underline{A}}\,\underline{\underline{U}}(\underline{\underline{A}}\,\underline{\underline{U}})^H\right)$

$$= \mathrm{Tr}(\underline{\underline{A}}\,\underbrace{\cancel{\underline{\underline{U}}\,\underline{\underline{U}}^H}}_{}\overset{I}{}\,\underline{\underline{A}}^H) = \|\underline{\underline{A}}\|_F^2 \checkmark$$

$$\|\underline{\underline{U}}\,\underline{\underline{A}}\|_F^2 = \|(\underline{\underline{U}}\,\underline{\underline{A}})^H\|_F^2 \quad (\text{by } (1))$$

$$= \mathrm{Tr}\left((\underline{\underline{U}}\,\underline{\underline{A}})^H\underline{\underline{U}}\,\underline{\underline{A}}\right)$$

$$= \mathrm{Tr}\left(\underline{\underline{A}}^H\,\cancel{\underline{\underline{U}}^H\overset{I}{\underline{\underline{U}}}}\,\underline{\underline{A}}\right)$$

$$= \mathrm{Tr}(\underline{\underline{A}}^H\underline{\underline{A}}) = \|\underline{\underline{A}}\|_F^2$$

(using (1))

iii) $\|\underline{\underline{A}}\|_F = \|\underline{\underline{U}}\,\underline{\underline{\Sigma}}\,\underline{\underline{V}}^H\|_F = \|\underline{\underline{\Sigma}}\|_F$

since $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are unitary matrices and the Frobenius norm is invariant under a unitary operation.

$$\Rightarrow \|\underline{\underline{A}}\|_F = \left(\sum \sigma_k^2\right)^{\frac{1}{2}}$$

iv) $\sigma_1 \leq \|A\|_F \leq \sqrt{\min(m,n)}\;\sigma_1$

When $\sigma_1 = \sigma_{max}$

c) If $\kappa_2(\underline{\underline{A}}) = 1 \Rightarrow \sigma_{max} = \sigma_{min}$, all singular values are equal (and non zero) $\therefore \underline{\underline{\Sigma}} = \sigma \underline{\underline{I}}$

$$\underline{\underline{A}} = \sigma \underline{\underline{U}}\,\underline{\underline{V}}^H$$

Note that $(\underline{\underline{U}}\,\underline{\underline{V}}^H)(\underline{\underline{U}}\,\underline{\underline{V}}^H)^H$

$$= \underline{\underline{U}}\,\underline{\underline{V}}^H \underline{\underline{V}}\,\underline{\underline{U}}^H = \underline{\underline{I}},$$

i.e. $\underline{\underline{U}}\,\underline{\underline{V}}^H$ is a unitary matrix

$\Rightarrow \underline{\underline{A}} = \sigma \underline{\underline{U}}\,\underline{\underline{V}}^H$ is a <u>scaled unitary matrix</u>

CRIB for

Question on Optimization

3M1

LENT 2021

MAB

Q3.

a

X_A = n° of A units produced

X_B = n° of B units produced

From description constraints are

$$12X_A + 25X_B \leq 30 \times 60$$ {assembly time}

$$-0.4X_A + X_B \geq 0$$ {from $x_B \geq \frac{2}{5}X_A$}

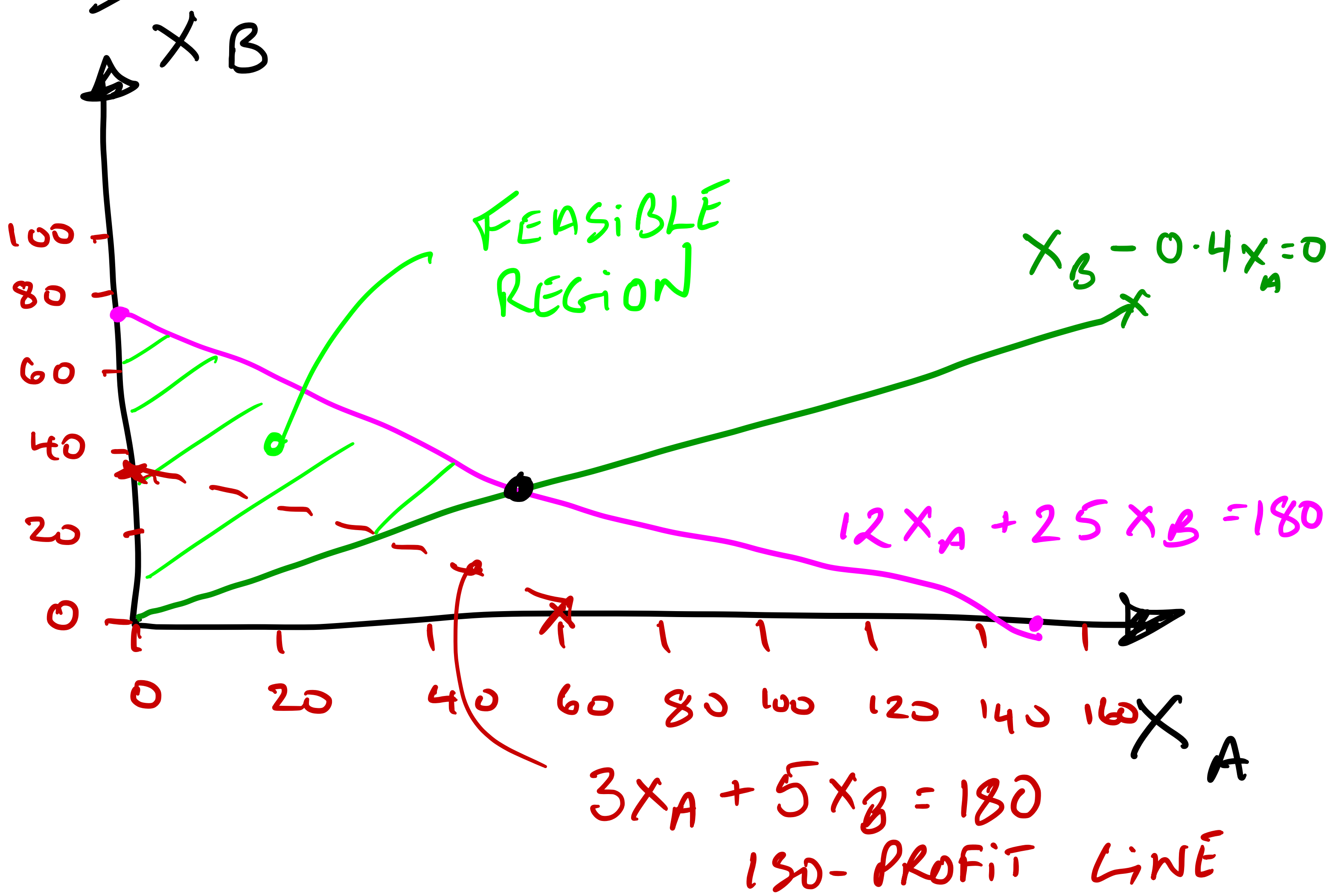
$$X_A, X_B \geq 0$$

OBJECTIVE is

$$\text{MAX} \quad 3X_A + 5X_B$$

20%

b.



$$12X_A + 25X_B = 1800$$

$$\frac{2}{5}X_A - X_B = 0 \quad \times 25$$

$$10X_A - 25X_B = 0$$

$$22X_A = 1800 \Rightarrow X_A = \frac{1800}{22} = 81.8$$

$$X_B = \frac{2}{5} \times \frac{1800}{22} = 32.7$$

$X_A = 81.8$, $X_B = 32.7$

$$3 \times 81.8 + 5 \times 32.7 \simeq £409$$

30%

c.

$$\text{MAX} \quad Z = 2x + 5y$$

s.t

$$3x + 5y \leq 8$$

$$2x + 7y \leq 12$$

$$x, y \geq 0$$

IN CANONICAL FORM

$$\text{MAX} \quad Z = 2x + 5y$$

s.t

$$3x + 5y + u = 8$$

$$2x + 7y + v = 12$$

$$x, y, u, v \geq 0$$

Change Max Z

to Min $-Z$

10%

D.

INITIAL TABLEAU

x	y	u	v	
3	5	1	0	8
2	7	0	1	12
-2	-5	0	0	0

10%

e.g.

x	y	u	v		
3	5	1	0	8	8/5 smallest
2	7	0	1	12	12/7
−2	−5	0	0	0	

most −ve

÷ by 5

x	y	u	v	
3/5	1	1/5	0	8/5
2	7	0	1	12
−2	−5	0	0	0

30%

$\boxed{\begin{array}{l} R2 - 7R1 \to R2 \\ R3 + 5R1 \to R3 \end{array}}$

x	y	u	v	
3/5	1	1/5	0	8/5
−11/5	0	−7/5	1	4/5
1	0	1	0	8

no −ve so done

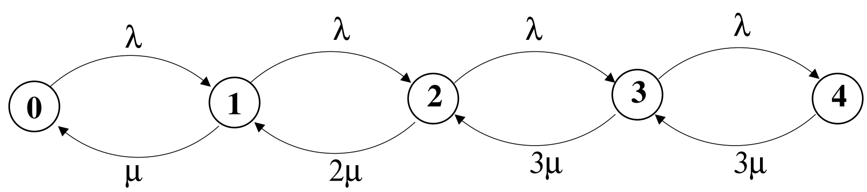
Basic $y = 8/5$ and $v = 4/5$ $z = 8$

$x = 0$

3M1 Mathematical Methods, 2021

1. *Birth Death Processed*

   (a) The state-space and transition rates are shown below.



[10%]

(b)(i) In time Δt for the probability of queue length 2 the following operations can occur in terms of the changes in the probability mass

- births and deaths from the current state yields: $-(\lambda + 2\mu)\Delta t \pi_2(t)$
- a death from queues of length 3 yields: $3\mu\Delta t\pi_3(t)$
- a birth from queues of length 1 yields: $\lambda\Delta t\pi_1(t)$

The assumptions behind this are that the probability of multiple events occurring is very small (ignored) as the size of Δt is very small.

Combining all of these together the probability mass associated with state 2

$$\pi_2(t + \Delta t) = (1 - \lambda\Delta t - 2\mu\Delta t)\pi_2(t) + \lambda\Delta t\pi_1(t) + 3\mu\Delta t\pi_3(t)$$

[15%]

(b)(ii) Taking the limit as $\Delta t \to 0$ yields the following form for $\mathbf{Q}$

$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ \mu & -\lambda-\mu & \lambda & 0 & 0 \\ 0 & 2\mu & -\lambda-2\mu & \lambda & 0 \\ 0 & 0 & 3\mu & -\lambda-3\mu & \lambda \\ 0 & 0 & 0 & 3\mu & -3\mu \end{bmatrix}$$

[20%]

(c)(i) At equilibrium the loss from state 1 is the same as the gain from state 1. hence

$$\begin{aligned}
0 &= -\lambda P_0 + \mu P_1, & P_1 &= \tfrac{\lambda}{\mu}P_0 \\
0 &= \lambda P_0 - (\lambda+\mu)P_1 + 2\mu P_2 & P_2 &= \tfrac{1}{2\mu}\left((\lambda+\mu)P_1 - \lambda P_0\right) = \tfrac{\lambda^2}{2\mu^2}P_0 \\
0 &= \lambda P_1 - (\lambda+2\mu)P_2 + 3\mu P_3 & P_3 &= \tfrac{1}{3\mu}\left((\lambda+2\mu)P_2 - \lambda P_1\right) = \tfrac{\lambda^3}{3\times 2\mu^3}P_0 \\
0 &= \lambda P_2 - (\lambda+3\mu)P_3 + 3\mu P_4 & P_4 &= \tfrac{1}{3\mu}\left((\lambda+3\mu)P_3 - \lambda P_2\right) = \tfrac{\lambda^4}{3^2\times 2\mu^4}P_0 \\
0 &= \lambda P_3 - 3\mu P_4 & P_4 &= \tfrac{\lambda}{3\mu}P_3
\end{aligned}$$

[30%]

(c)(ii) The distribution over the queue length must satisfy a sum to one constraint. So

$$\left(1 + \sum_{i=1}^{4} k_i \left(\frac{\lambda}{\mu}\right)^i\right) P_0 = 1$$

Thus

$$P_0 = \frac{1}{\left(1 + \sum_{i=1}^{4} k_i \left(\frac{\lambda}{\mu}\right)^i\right)}$$

[15%]

(c) If customers will join the queue then the maximum length is inf, changing the summation. The form of recursion will carry on, resulting in the form of recursion if a stationary distribution is reached

$$P_i = \frac{\lambda}{3\mu} P_{i-1}$$

If this recursion grows then the system will not hit a steady state. [10%]

Engineering Tripos Part IIA 2021

Assessors Report, Module 3M1

The examination was taken by 153 candidates of whom 142 with Part 1A., plus 13 students with no Part IA.

A small degree of scaling was required, achieved using the scaling tool to map the boundaries as follows: 46/60(raw) to 42/60(scaled); 35/60(raw) to 36/60(scaled); 26/60(raw) to 30/60(scaled).

Q1. **Linear Algebra** Part (a) is straightforward and was well answered by most candidates. A small number of candidates did not realise that a norm must be real-valued. Part (b)(i) was generally well-answered, although some candidates re-stated the question text without convincing steps to demonstrate that they knew how to arrive at the result, or considered a single size matrix only rather than the general case. For Part(b)(ii), most found the proof for |**AU**|_F easy, but many complicated or did not fully justify the |**UA**|_F case. The simple approach was to note that |**A**|_F = |**A**^H|_F, which follows trivially from the definition of the Frobenius norm. Part(b)(iii) was well answered by some by using the fact that the Frobenius norm is invariant under a unitary operation, as given earlier in the question. Others overcomplicated the proof, and used other results without proof rather than results proved in the question. For Part(b)(iv), as surprising number of candidates gave only a lower or upper bound when the question asked for both. For Part(c), nearly all candidates identified that the ratio of the maximum to the minimum singular value must be 1 and/or that all singular values must be equal and nonzero. Some identified that this implied **A** = \sigma S**UV**^H, where S is the singular value. Very few identified **A** as a scaled unitary matrix.

Q2. **Optimisation** This question examined constrained linear optimisation from a graphical construction and by the Simplex method using a tableau. Both parts of the question were answered well with no issues in the first part where a description of the problem needed to be interpreted and written as a constrained optimisation problem. The graphical solution of this problem highlighted the difficulty some students had in interpreting the constraints and defining the feasible region for solutions. However overall, this was answered straightforwardly. The second part of the question tested how students might address the problem of inequality constraints with the use of slack variables, followed by the construction of a tableau and the simplex method. Again, there were no overall issues with this part of the question beyond arithmetic slips

Q3. **Birth-Death Processes** This question examined birth-death processes when applied to a queuing system. Though a number of students gave very good answers to this question, there were also a large number of poor answers. Some students treated the process as discrete, rather than continuous, in time despite the equations given in the question. Few students spotted the need requirements for the process to reach steady state for part (c). with a few students demonstrating a high degree of rigour in showing that all of the functions in the sequence are in L2, quite impressive.

M.A.Girolami (Principal Assessor)