Version GTP/3

EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 6 May 2014 2 to 3.30

Module 3M1

MATHEMATICAL METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3M1 data sheet (4 pages)
Engineering Data Book

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Explain what is meant by the *pseudo inverse*, $\mathbf{A}^{+}$, of a matrix **A**, and the role that it plays in finding the optimal least-squares solution to problems of the form $\mathbf{A}\,\underline{x} = \underline{b}$. [15%]

(b) Using the method you have described in (a), find the least-squares solution of

$$\mathbf{A}\,\underline{x} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & -0.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \\ 1 \end{bmatrix}$$ [50%]

(c) Determine the l_2 norm of the pseudo inverse, $\mathbf{A}^{+}$, of the matrix **A** used in (b). [15%]

(d) Find a non-zero vector $\underline{x}$, such that

$$\|\mathbf{A}^{+}\underline{x}\| = \|\mathbf{A}^{+}\|\,\|\underline{x}\|$$ [20%]

2 Figure 1 shows two frictionless rigid bodies (carts) A and B connected by three linear elastic springs with spring constants k_1, k_2 and k_3. When the applied force P is zero, the springs are at their natural (unstretched) lengths.

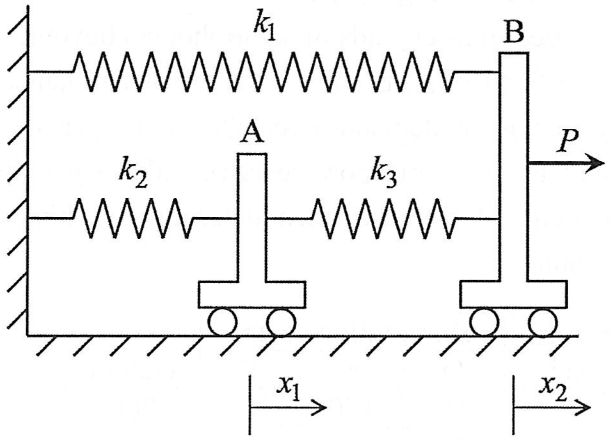

Fig. 1

According to the principle of minimum potential energy, the system will be in equilibrium under the load P when the potential energy is minimised. The potential energy U of the system is given by:

$$U = \tfrac{1}{2}k_2 x_1^2 + \tfrac{1}{2}k_3(x_2 - x_1)^2 + \tfrac{1}{2}k_1 x_2^2 - P x_2$$

(a) Using appropriate optimality conditions, find general expressions for the equilibrium extensions x_1^* and x_2^* in terms of k_1, k_2, k_3 and P. [30%]

(b) Show that Newton's Method will converge on this potential energy minimum in one iteration from any initial solution (x_1, x_2). Is this behaviour to be expected? [30%]

(c) For the specific case where $k_1 = 1$, $k_2 = 2$, $k_3 = 3$ and $P = 1$, execute two iterations of the Conjugate Gradient Method from an initial solution $(x_1, x_2) = (0, 0)$ and comment on the behaviour observed. [40%]

3 The Cheapo Smartphone Company manufactures smartphones in three factories located in Altrincham (A), Baldock (B) and Cardiff (C). The factories also serve as distribution centres, satisfying local demand, and the company has additional distribution centres in Derby (D), Edinburgh (E) and Farnborough (F). Estimates have been made of the output (measured in truckloads of smartphones) beyond that needed to satisfy local demand from each factory (A, B and C) for the next financial year and of the requirements to satisfy customer demand over the same period of the other distribution centres (D, E and F). The transport costs (in arbitrary units) from each factory to each distribution centre have also been determined. This information is summarised in the following table:

Transport cost per truckload		Distribution centre D	E	F	Output
Factory	A	50	180	160	200
	B	80	300	60	200
	C	125	310	100	300
Demand		150	250	300	

The managing director has asked you to find the allocation of factory outputs to distribution centres that minimises the cost of transporting the smartphones.

(a) Explain what is meant by the *minimum matrix method* for finding an initial feasible basis for a transportation problem, and show that the initial basis of this transportation problem given by this method is that shown in Fig. 2. [20%]

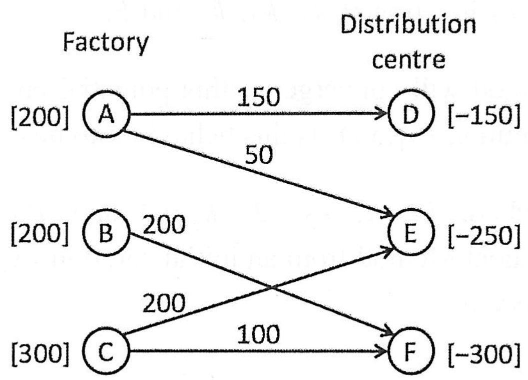

Fig. 2

 (cont.

(b) Starting from the initial feasible basis given in (a), use the *simplex method for transportation problems* to find the optimal allocation of factory outputs to distribution centres. You are reminded that, for such problems, the simplex multipliers for supply nodes u_i and demand nodes v_j in the current basis are related by $u_i + v_j = c_{ij}$, where c_{ij} is the transport cost per unit between the nodes, and that the reduced costs for non-basic variables are given by $\bar{c}_{ij} = c_{ij} - u_i - v_j$. [50%]

(c) Explain what is meant by the *northwest-corner method* for finding an initial feasible basis for a transportation problem. Use it to find an alternative initial feasible basis for this transportation problem. Why is the minimum matrix method generally preferred in practice? [30%]

4 (a) The $N \times N$ matrix $\mathbf{P}$ governs transitions in a finite state-space homogeneous Markov chain by

$$\text{Prob}(X_{n+1} = k \mid X_n = j) = p_{jk} = j,k^{\text{th}} \text{ element of } \mathbf{P}$$

Explain why the sum of the elements in any row of $\mathbf{P}$ is equal to 1. [10%]

(b) By considering the sum of the first $N-1$ columns of $\mathbf{P}-\mathbf{I}$, where $\mathbf{I}$ is the identity matrix, or otherwise, show that $\mathbf{P}$ has at least one eigenvalue equal to unity. [20%]

(c) A 3-state Markov process is governed by transition probabilities as shown in the transition diagram, Fig. 3. If an individual undergoing this process starts in state 1, find the expected number of transitions before the individual is next in state 1. [40%]

(d) Find the expected number of transitions before the individual is next in state 1 having passed through state 3. [15%]

(e) If a large number of individuals are undergoing this process, show that the proportions that one would expect to find in states 1, 2 and 3, once the process has reached equilibrium, are respectively:

$$\frac{1}{5-4a} : \frac{2-2a}{5-4a} : \frac{2-2a}{5-4a}$$ [15%]

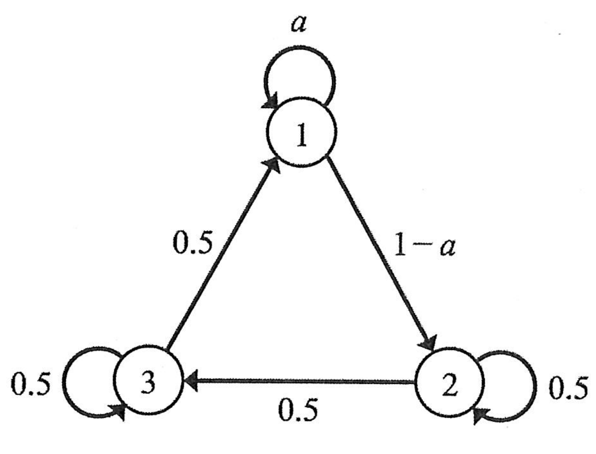

Fig. 3

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