EGT2
ENGINEERING TRIPOS PART IIA

Monday 3 May 2021   1.30 to 3.10

Module 3M1

MATHEMATICAL METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet and at the top of each answer sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed.
You are allowed access to the electronic version of the Engineering Data Books.

10 minutes reading time is allowed for this paper at the start of the exam.

The time taken for scanning/uploading answers is 15 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

1 (a) Compute the l_1, l_2 and l_∞ norms of the following vectors:

(i) $\boldsymbol{x} = \begin{bmatrix} 1 & 1-i & 1+i \end{bmatrix}$ [10%]

(ii) $\boldsymbol{x} \in \mathbb{C}^n$ where the rth component is $x_r = \sqrt{(-1)^r}$ [10%]

(b) The Frobenius norm of a matrix $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ is defined as $\|\boldsymbol{A}\|_F^2 := \sum_{i,j} |A_{ij}|^2$.

(i) Show that $\|\boldsymbol{A}\|_F = \sqrt{\mathrm{T}r\left(\boldsymbol{A}\boldsymbol{A}^H\right)}$. The trace of square matrix $\boldsymbol{B}$ is defined as the sum of its diagonal values, i.e. $\mathrm{T}r\boldsymbol{B} = \sum_i B_{ii}$. [10%]

(ii) Show that the Frobenius norm is invariant under unitary operations, i.e. $\|\boldsymbol{A}\|_F = \|\boldsymbol{U}\boldsymbol{A}\|_F = \|\boldsymbol{A}\boldsymbol{U}\|_F$ where $\boldsymbol{U}$ is a unitary matrix. [20%]

(iii) Show that $\|\boldsymbol{A}\|_F = \sqrt{\sum_k \sigma_k^2}$, where $\{\sigma_k\}$ are the singular values of $\boldsymbol{A}$. [20%]

(iv) Give upper and lower bounds for $\|\boldsymbol{A}\|_F$ terms of the maximum singular value of $\boldsymbol{A}$. [10%]

(c) If the condition number is $\kappa_2(\boldsymbol{A}) = 1$, where $\boldsymbol{A}$ is a square matrix, by considering the SVD of $\boldsymbol{A}$ what can you say about the properties of $\boldsymbol{A}$? [20%]

2 A company manufactures two products (A and B) and the profit per unit sold is £3 and £5 respectively. Each product has to be assembled on a particular machine, each unit of product A taking 12 minutes of assembly time and each unit of product B 25 minutes of assembly time. The company estimates that the machine used for assembly has an effective working week of only 30 hours (due to maintenance and breakdown). Technological constraints mean that for every five units of product A produced at least two units of product B must be produced.

(a) Formulate the problem of how much of each product to produce as a linear program. [20%]

(b) By solving this linear program graphically specify the number of units of product A and B that should be produced to maximise profit and giving the value of that profit. [30%]

(c) For the linear programming problem

$$\begin{aligned} &\text{Maximise} \quad z = 2x + 5y \\ &\text{subject to} \\ &\quad 3x + 5y \leq 8 \\ &\quad 2x + 7y \leq 12 \\ &\quad x \geq 0, y \geq 0 \end{aligned}$$

(i) Rewrite the linear programme in canonical form. [10%]

(ii) Construct the initial tableau of the canonical form. [10%]

(iii) With the initial tableau use the simplex method to obtain the optimal value of the objective function and the corresponding values of x and y. [30%]

3 The total number of customers in a checkout process, either in the queue or being served at checkouts, is described by a birth-death process. λ is the rate of customers entering the process, the *birth rate*, per unit time and μ is the rate at which a customer is served by a single checkout, the *death rate*, per unit time. There are 3 checkouts available to serve the queue so the maximum "death" rate is 3μ. Customers refuse to join the queue when it is of length one, so the maximum number of customers in the process is four.

(a) Draw the state-space associated with the number of customers in the process and the transition rates between these states. [10%]

(b) The change in the probability distribution of the number of customers in the process can be described by the following equation

$$\frac{d\boldsymbol{\pi}(t)}{dt} = \boldsymbol{\pi}(t)\mathbf{Q}$$

(i) Show that the following expression is satisfied for the change in the probability of two customers in time Δt

$$\pi_2(t + \Delta t) = (1 - \lambda\Delta t - 2\mu\Delta t)\pi_2(t) + \lambda\Delta t\pi_1(t) + 3\mu\Delta t\pi_3(t)$$

where $\pi_i(t)$ is the i-th element of row vector $\boldsymbol{\pi}(t)$. You should make it clear any assumptions that are made in this derivation. [15%]

(ii) Hence find the elements of the transition rate matrix $\mathbf{Q}$. [20%]

(c) The steady state distribution of the number of customers in the process is given by $[P_0, \ldots, P_4]$. This will occur when

$$\frac{d\boldsymbol{\pi}(t)}{dt} = \mathbf{0}$$

(i) Show that for $0 < i \leq 4$

$$P_i = k_i \left(\frac{\lambda}{\mu}\right)^i P_0$$

You should clearly give the values of k_i. [30%]

(ii) Using the expression in (c)(i) find the value of P_0 in terms of μ, λ and k_i. [15%]

(d) If customers are willing to join the queue however long it is, how would this change your answer to part (c)(i)? [10%]

END OF PAPER