3C7/2024

Q1

(a) With $\tau_{xy}=0$ the equilibrium equations reduce to

$$\frac{\partial \sigma_{xx}}{\partial x} = 0 \quad \& \quad \frac{\partial \sigma_{yy}}{\partial y} = 0$$

$$\Rightarrow \quad \sigma_{xx} = f(y) \quad \& \quad \sigma_{yy} = g(x)$$

However, $\sigma_{yy} = 0$ on $y = 0 \ \forall x$

$$\Rightarrow g(x) = 0, \text{ ie } \sigma_{yy} = 0$$

The compatibility equation $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx}+\sigma_{yy}) = 0$

then implies $\frac{\partial^2 f}{\partial y^2} = 0$

$$\Rightarrow \quad f = -p_0 - p_1 y = \sigma_{xx}$$

But $\sigma_{xx} = -p(y)$ on $x = 0$

$$\Rightarrow \quad p(y) = p_0 + p_1 y$$

(b) If $\tau = 0$ in all directions, Mohr's circle reduces to a point with $\sigma_{xx} = \sigma_{yy} = \sigma_0$.

Equilibrium equations then mean $\dfrac{\partial \sigma_{xx}}{\partial x} = \dfrac{\partial \sigma_0}{\partial x} = 0$

& $\dfrac{\partial \sigma_{yy}}{\partial y} = \dfrac{\partial \sigma_0}{\partial y} = 0$

ie σ_0 is spatially uniform. Thus, the boundary pressures consistent with this uniform field are $q(x) = p(y) = -\sigma_0$.

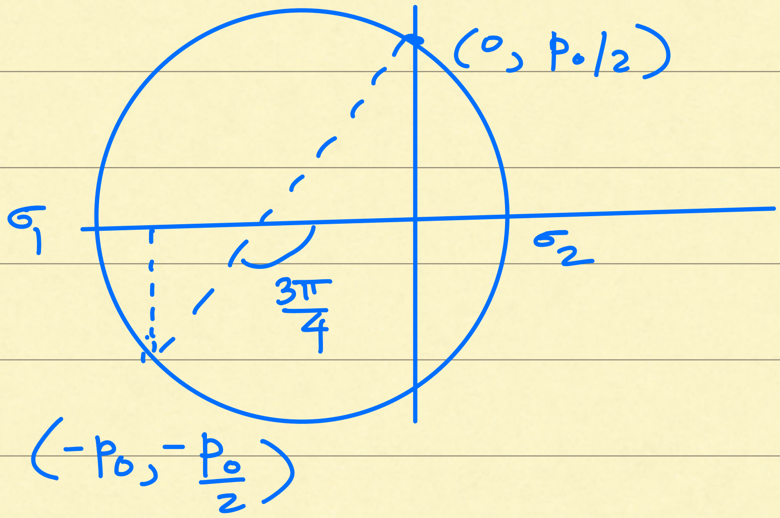
(c)(i) With $p(y) = p_0$ & $q(x) = 0$ from (a) $\sigma_{xx} = -p_0$ & $\sigma_{yy} = 0$. Superpose a uniform stress $\tau_{xy} = p_0/2$ on $x=0$ & $y=0$. The equilibrium equations become

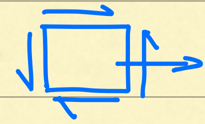
$$\frac{\partial \tau_{xy}}{\partial x} = \frac{\partial \tau_{xy}}{\partial y} = 0 \Rightarrow \tau_{xy} = \frac{p_0}{2}.$$

Combining with (a) at any point (x,y)

$$(\sigma_{xx}, \sigma_{yy}, \tau_{xy}) = (-p_0, 0, p_0/2)$$

(ii)





$$\sigma_1 = -\frac{p_0}{2} - \sqrt{\left(\frac{p_0}{2}\right)^2 + \left(\frac{p_0}{2}\right)^2} = -\frac{p_0}{2}\left(\sqrt{2}+1\right)$$

at $\pi/8$ clockwise from x-direction

$$\sigma_2 = -\frac{p_0}{2} + \sqrt{\left(\frac{p_0}{2}\right)^2 + \left(\frac{p_0}{2}\right)^2} = \frac{p_0}{2}\left(\sqrt{2}-1\right)$$

at $3\pi/8$ anticlockwise from x-direction.

Q2

(a) $\phi = Ar^2\theta^m$

$$\frac{\partial\phi}{\partial r} = 2Ar\theta^m \quad ; \quad \frac{\partial^2\phi}{\partial r^2} = 2A\theta$$

$$\frac{\partial^2\phi}{\partial\theta^2} = m(m-1)Ar^2\theta^{m-2}$$

$$\nabla^2\phi = \left[4\theta^2 + m(m-1)\right]A\theta^{m-2}$$

$$\nabla^4\phi = \left[4m(m-1)\theta^2 + m(m-1)(m-2)(m-3)\right]A\theta^{m-4} = 0$$

$\Rightarrow \quad m = 0, 1$

(b) (i)

Stresses for $\phi = Ar^2\theta$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 2A\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = 2A\theta$$

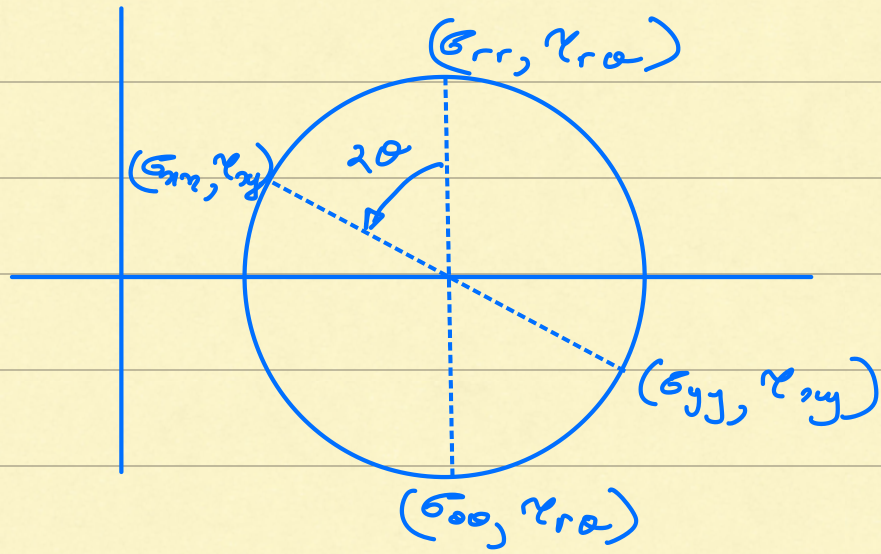
$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = -A$$

$\Rightarrow$ Tractions on surface of half-space are

$\sigma_{\theta\theta}(r, 0) = 0 \qquad \tau_{r\theta}(r, 0) = -A$

$\sigma_{\theta\theta}(r, \pi) = 2\pi A \qquad \tau_{r\theta}(r, \pi) = -A$

(ii)

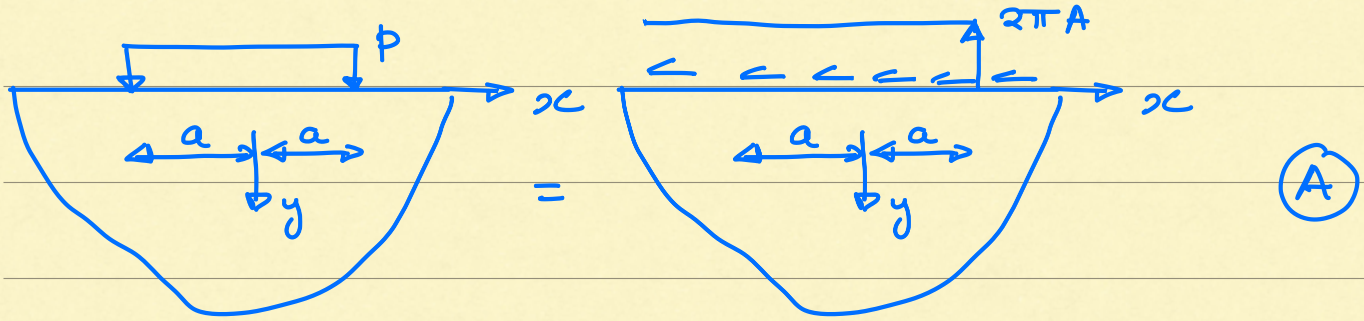


$$\sigma_{xx} = A(2\theta - \sin 2\theta) = A\left[2\tan^{-1}\frac{y}{x} - \frac{2xy}{x^2+y^2}\right]$$

$$\sigma_{yy} = A(2\theta + \sin 2\theta) = A\left[2\tan^{-1}\frac{y}{x} + \frac{2xy}{x^2+y^2}\right]$$

$$\tau_{xy} = -A\cos 2\theta = -A\left(\frac{x^2-y^2}{x^2+y^2}\right)$$

(iii)



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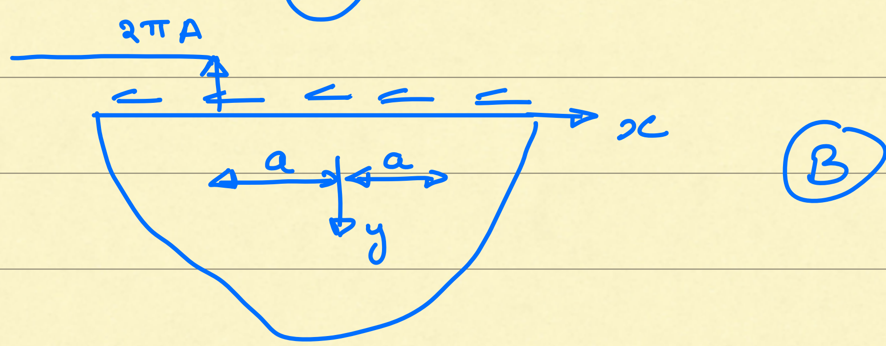


A $\qquad \tau_{xy} = -A \dfrac{(x-a)^2 - y^2}{(x-a)^2 + y^2}$

B $\qquad \tau_{xy} = -A \dfrac{(x+a)^2 - y^2}{(x+a)^2 + y^2}$

Solution = A − B

B.C $\Rightarrow \quad \sigma_{yy}(x,0) = -p = 2\pi A$

$$A = -\frac{p}{2\pi}$$

$$\Rightarrow \tau_{xy} = \frac{p}{2\pi}\left[\frac{(x-a)^2 - y^2}{(x-a)^2 + y^2} + \frac{(x+a)^2 - y^2}{(x+a)^2 + y^2}\right]$$

Q3

(a) $\phi = 0$ on boundary is automatically satisfied by boundary equation

$$x^4 - 6x^2y^2 + y^4 + 5c^2(x^2+y^2) - 6c^4 = 0$$

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi =$$

$$\left[12x^2 - 12y^2 - 12x^2 + 12y^2 + 10c^2 + 10c^2\right] = 20\beta c^2$$

$\Rightarrow$ with $-2G\alpha = 20\beta c^2$ ie $\beta = -\dfrac{G\alpha}{10c^2}$

ϕ is a valid Prandtl stress function.

(b)

$$\tau_{xz} = \frac{\partial\phi}{\partial y} = \beta\left[-12x^2y + 4y^3 + 10c^2y\right]$$

$$= -\frac{G\alpha}{10c^2}\left[-12x^2y + 4y^3 + 10c^2y\right]$$

$$\tau_{yz} = -\frac{\partial\phi}{\partial x} = -\beta\left[4x^3 - 12xy^2 + 10c^2x\right]$$

$$= \frac{G\alpha}{10c^2}\left[4x^3 - 12xy^2 + 10c^2x\right]$$

on $x = y$; $\tau_{xz} = -\tau_{yz} = \frac{G\alpha}{10c^2}\left[8x^3 - 10c^2x\right]$

Thus the shear stress component along $x = y$ which is given by

$$\frac{1}{\sqrt{2}}\tau_{xz} + \frac{1}{\sqrt{2}}\tau_{yz} = 0$$ as expected by

symmetry.

At $(x, y) = (c, c)$ $\tau_{xz} = -\tau_{yz} = \frac{G\alpha}{10c^2}\left(8c^3 - 10c^3\right)$

$$= -0.2G\alpha c$$

For a perfectly square cross-section $\tau_{xz} = \tau_{yz} = 0$ at the corner as complementary values on the longitudinal surfaces are zero.

(c) On $y = 0$ $\tau_{yz} = \frac{G\alpha}{10c^2}\left(4x^3 + 10c^2x\right)$

which is a max at $x = c$

ie $\quad \tau_{yz}\big|_{max} = \dfrac{G\alpha}{10c^2}\left(14c^3\right) = 1{\cdot}4\, G c \alpha.$

(d)

$$Q = 2\int_{-c}^{c}\int_{-c}^{c} \phi \, dx\, dy$$

$$= -\frac{2G\alpha}{10c^2}\int_{-c}^{c}\int_{-c}^{c}\left[x^4 - 6x^2y^2 + 5c^2(x^2+y^2) - 6c^4\right] dx\, dy$$

$$= -\frac{2G\alpha}{10c^2}\int_{-c}^{c}\left[\frac{x^5}{5} - \frac{6x^2y^2}{3} + y^4x + \frac{5c^2x^3}{3} + 5c^2y^2x - 6c^4x\right]_{x=-c}^{x=c} dy$$

$$= -\frac{2G\alpha}{10c^2}\int_{-c}^{c}\left(\frac{2c^5}{5} - 4c^3y^2 + 2cy^4 + \frac{10c^5}{3} + 10c^3y^2 - 12c^5\right) dy$$

$$= -\frac{2G\alpha}{10c^2}\left[\frac{2c^5y}{5} - \frac{4c^3y^3}{3} + \frac{2cy^5}{5} + 10c^5y + 10c^3y - 12c^5y\right]_{y=-c}^{y=c}$$

$$Q = -\frac{2G\alpha c^4}{10}\left(-\frac{176}{15}\right) = \frac{176}{75}G\alpha c^4$$

4 a) Lame's equations

$$\sigma_{rr} = A - \frac{B}{r^2} \qquad \sigma_{\theta\theta} = A + \frac{B}{r^2}$$

$r \to \infty \qquad \sigma_{rr} = A = \lambda$

$r = a \qquad \sigma_{rr} = \lambda - \frac{B}{a^2} = \frac{\lambda}{5} \qquad \Rightarrow B = \frac{4a^2\lambda}{5}$

$$\sigma_{rr} = \lambda - \frac{4}{5}\frac{a^2}{r^2}\lambda = \lambda\left(1 - \frac{4}{5}\frac{a^2}{r^2}\right)$$

$$\sigma_{\theta\theta} = \lambda + \frac{4}{5}\frac{a^2}{r^2}\lambda = \lambda\left(1 + \frac{4}{5}\frac{a^2}{r^2}\right)$$

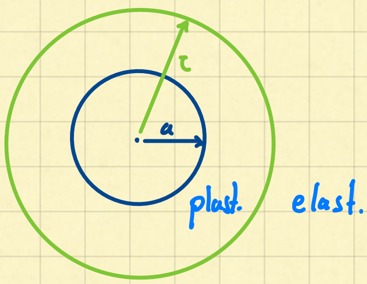
b) Note that $\sigma_{\theta\theta} > \sigma_{rr} > \sigma_{zz} = 0$

At the hole boundary with $r = a$ according to Tresca:

$\sigma_{\theta\theta} - 0 = Y \Rightarrow \lambda = \frac{5r^2 Y}{4a^2 + 5r^2}$ with $r = a \Rightarrow \lambda = \frac{5Y}{9}$

c) Equilibrium equation the same whether elastic or plastic

$$\frac{d}{dr}(r\sigma_{rr}) = \sigma_{\theta\theta}$$



In the plastic region: $\sigma_{\theta\theta} = Y$

$$\frac{d}{dr}(r\sigma_{rr}) = Y$$

$$\sigma_{rr} = Y + \frac{D}{r}$$

Hole boundary

$r = a \qquad \sigma_{rr} = Y + \frac{D}{a} = \frac{\lambda}{5} \qquad \Rightarrow \qquad D = a\left(\frac{\lambda}{5} - Y\right)$

$$\Rightarrow \sigma_{rr} = Y + \frac{a}{r}\left(\frac{\lambda}{5} - Y\right)$$

Plastic - elastic interface

Plastic inner sheet

$$p_i = \sigma_{rr}^{p} = Y + \frac{a}{c}\left(\frac{\lambda}{5} - Y\right)$$

Elastic outer sheet

$$\sigma_{rr}^{e} = A - \frac{B}{r^2} \qquad \sigma_{\theta\theta}^{e} = A + \frac{B}{r^2}$$

$$r \to \infty \qquad \sigma_{rr}^{e} = A = \lambda$$

$$r = c \qquad \sigma_{rr}^{e} = \lambda - \frac{B}{c^2} = p_i \implies B = c^2(\lambda - p_i)$$

$$\sigma_{\theta\theta}^{e} = \lambda + \frac{c^2}{r^2}(\lambda - p_i)$$

Onset of plasticity

$$Y = \lambda + \lambda - p_i \implies p_i = 2\lambda - Y$$

$$\implies 2\lambda - Y = Y + \frac{a}{c}\left(\frac{\lambda}{5} - Y\right)$$

$$c = \frac{a(5y - \lambda)}{10(y - \lambda)}$$