Q 1)

(a)
Fixed left end:

$$u(x=0) = 0$$
$$\frac{du}{dx}|_{x=0} = 0$$

Moment and stress free right end:

$$\frac{d^2u}{dx^2}|_{x=L} = 0$$
$$\frac{d^3u}{dx^3}|_{x=L} = 0$$

(b)

$$\int_0^L EI\frac{d^2u}{dx^2}\frac{d^2v}{dx^2}dx = \int_0^L f(x)vdx + EI\frac{d^2u}{dx^2}\frac{dv}{dx}|_0^L - EI\frac{d^3u}{dx^3}v|_0^L$$

Inserting boundary condition:

$$\int_0^L EI\frac{d^2u}{dx^2}\frac{d^2v}{dx^2}dx = \int_0^L q(x)vdx$$

(c) Shape functions with continuous first derivative is requried. One can use the shear deformable theory where the beam rotation is no longer equal to the derivative of the deflection. There are at most first-order derivatives in the weak form. Therefore simple Lagrange finite element shape functions with discontinuous first derivatives can be used.

(d)
Let u_h denote the finite element solution. $m = EI\frac{d^2u_h}{dx^2}$ while $q = -EI\frac{d^3u_h}{dx^3}$.
The moment is more accurate. From the a-priori error estimate

$$||u - u_h||_2 \propto \mathcal{O}(h^2)$$

while

$$||u - u_h||_3 \propto \mathcal{O}(h).$$

Therefore the error in m is smaller.

Q2) a)

$$\nabla^2 T + cT + s = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + s = 0$$

$$\int_\Omega \frac{\partial^2 T}{\partial x^2} w = \int_\Omega \left(\frac{\partial T}{\partial x} w \right)' - \int_\Omega \frac{\partial T}{\partial x} \frac{\partial w}{\partial s} = \underbrace{\cancel{\int_\Gamma \frac{\partial T}{\partial x} w n_x}}_{=0} - \int_\Omega \frac{\partial T}{\partial x} \frac{\partial w}{\partial x}$$

$$\Longrightarrow \int_\Omega \frac{\partial T}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial T}{\partial y} \frac{\partial w}{\partial y} - c \int_\Omega T w = \int_\Omega s w$$

b)

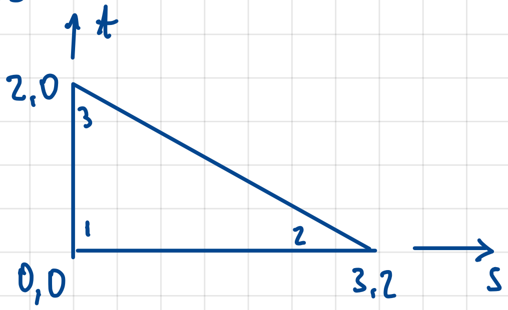
$$\frac{\partial T}{\partial x} \cdot n_x + \frac{\partial T}{\partial y} n_y + \beta T = \bar{q}$$

$$\int_\Gamma \left(\frac{\partial T}{\partial x} \cdot n_x + \frac{\partial T}{\partial y} n_y \right) w = \int_\Gamma (\bar{q} - \beta T) w$$

$$\Longrightarrow$$

$$\int_\Omega \frac{\partial T}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial T}{\partial y} \frac{\partial w}{\partial y} - c \int_\Omega T w + \beta \int_\Gamma T w = \int_\Omega s w - \int_\Gamma \bar{q} w$$

c)



$$N_2(\xi, \eta) = \frac{s}{3} \qquad N_3(\xi, \eta) = \frac{t}{2}$$

$$\nabla N_2 = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} \qquad \nabla N_3 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

Contrib. of $\int_\Omega \frac{\partial T}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial T}{\partial y}\frac{\partial w}{\partial y}$

$$\int \frac{\partial N_2}{\partial s}\frac{\partial N_2}{\partial s} + \frac{\partial N_2}{\partial t}\frac{\partial N_2}{\partial t} = 3\,\frac{1}{9} = \frac{1}{3}$$

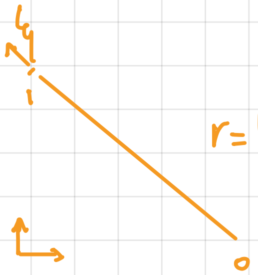
$$\int \frac{\partial N_2}{\partial s}\frac{\partial N_3}{\partial s} + \frac{\partial N_2}{\partial t}\frac{\partial N_3}{\partial t} = 3\cdot 0 = 0$$

Contrib. of $c\int_\Omega T w$

$$c\int_\Omega N_2 N_2 = \int_\Omega \frac{s^2}{9} = c\int_0^3 \frac{s^2}{9}\int_0^{2-\frac{2}{3}s} dt\,ds = c\int_0^3 2\frac{s^2}{9}\left(1-\frac{1}{3}s\right) = \frac{1}{2}c$$

$$c\int_\Omega N_2 N_3 = c\int_\Omega \frac{st}{6} = c\int_0^3 s\int_0^{2-\frac{2}{3}s} t\,dt\,ds = \frac{3}{2}c$$

Contrib. of $\beta\int_\Gamma Tw$



$r = \begin{pmatrix} 3-3\xi \\ 2\xi \end{pmatrix}$ $\quad r' = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$\sqrt{(r')^2} = \sqrt{13}$

$$\beta\int_\Gamma N_2 N_2 = \int_\Gamma \frac{s^2}{9}$$

$$= \beta\int_\Gamma (3-3\xi)^2\sqrt{13}\,d\xi = 10{,}82\,\beta$$

$$\beta\int N_2 N_3 = \int_\Gamma \frac{1}{6}\,2\xi(3-3\xi)\,d\xi =$$

Note that node 2 receives contrib. from two elements. Hence, the (2,2) contribs on previous page must be multiplied by two.

Q3)

a) Isoparametric mapping needed for defining shapefunctions for arbitrary quadrilaterals and curved boundaries.

b) i)
$$x(\xi,\eta) = 4.0 \cdot N_2(\xi,\eta) + 8.0\, N_3(\xi,\eta) + 4.0\, N_4(\xi,\eta)$$

$$y(\xi,\eta) = 5.0 \cdot N_3(\xi,\eta) + 5.0\, N_4(\xi,\eta)$$

	$\partial/\partial\xi$	$\partial/\partial\eta$
N_2	$\frac{1}{4}(1-\eta)$	$-\frac{1}{4}(1+\xi)$
N_3	$\frac{1}{4}(1+\eta)$	$\frac{1}{4}(1+\xi)$
N_4	$-\frac{1}{4}(1+\eta)$	$\frac{1}{4}(1-\xi)$

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 8 & 0 \\ 8 & 10 \end{pmatrix}$$

ii) $$J^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{2}{5} & \frac{2}{5} \end{pmatrix}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$u_x = \frac{1}{8}(1-\xi)(1+\eta)$$

$$u_y = \frac{1}{20}(1-\xi)(1+\eta)$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{2}{5} & \frac{2}{5} \end{pmatrix}$$

$$\varepsilon_{yy} = +\frac{1}{20}(1+\eta)\frac{2}{5} + \frac{1}{20}(1-\xi)\cdot\frac{2}{5} = \frac{1}{50}(2+\eta-\xi)$$

$$2\varepsilon_{xy} = \frac{\partial u_x}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial u_y}{\partial \xi}\frac{\partial \xi}{\partial x} + \cancel{\frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial x}}$$

$$= +\frac{1}{8}(1+\eta)\frac{2}{5} + \frac{1}{8}(1-\xi)\frac{2}{5} - \frac{1}{20}(1+\eta)\frac{1}{2}$$

$$= \frac{1}{40}(3+\eta-2\xi)$$

c) Triangle:

$$N_1 = 2(1-\xi-\eta)^2 - (1-\xi-\eta)$$

Quad:

$$N_5 = \frac{1}{2}(1-\xi)(1-\eta^2) - \frac{1}{2}(1-\xi^2)(1-\eta^2)$$

$$= \frac{1}{2}(\xi^2-\xi)(1-\eta^2)$$

Q 4)

(a)

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(b)

$$M_c = \int_0^L \rho A N^1(x) N^2(x) dx$$

where

$$N^1(x) = x/L, x \in [0, L] N^2(x) = 1 - x/L, x \in [0, L]$$

Hence M_c is given by

$$\rho A L \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

(c) Let ω be the eigen frequencies

$$\det(K - \omega^2 M) = 0$$

indicates

$$(EA - \frac{m}{3}\omega^2)^2 - (EA + \frac{m}{6}\omega^2)^2 = 0,$$

where and the mass of the beam $m = \rho A L$.
Therefore there are 2 eigen frequencies where $\omega_1 = 0$ and $\omega_2 = \sqrt{\frac{12EA}{mL}} = \sqrt{\frac{12E}{\rho}}\frac{1}{L}$. The corresponding eigenmodes follow from insertion of the eigenfrequencies into the eigenvalue problem, giving

$$U_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$U_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(d) The stability constraint requires $\triangle t \leq \frac{2}{\omega_{max}}$. Therefore $\triangle t_{cr} = \frac{2}{\omega_2} = \sqrt{\frac{\rho}{3E}} L$