Engineering Tripos Part IIA: Module 3C2
Manufacturing Engineering Tripos Part I: Paper P4B
Materials process modelling and failure analysis
CRIB - 2005/6 (M Sutcliffe)

1. (a) At the tip of the crack the theoretical elastic stress field has a square root singularity with $\sigma \propto 1/\sqrt{r}$. In practice materials cannot sustain the infinite stresses, and there is a zone near the tip of the crack with inelastic processes. These process absorb energy and contribute to the energy required for crack growth and hence the toughness. Typical mechanisms are plasticity, cracking, void growth, cleavage. Approaching the crack tip, the "local stress" reaches a stress σ_{ry} a distance r_y from the crack tip, and plastic flow occurs, for metals. Closer to the crack tip, the local stress remains at σ_y so we have defined a constant-stress region in which plastic deformation can occur. This is the plastic zone (or more generally the process zone) at the crack tip, which is approximated to a circle radius r_y centred on the crack tip.
Plane Stress: (Thin material): No stress out-of-plane,
Plane Strain: (Thick material): No strain out-of-plane

With plane strain the constraint of the surrounding material prevents out-of-plane deformation at the process zone. This builds up out-of plane stress so making deformation more difficult and reducing the region over which the stresses are high enough to cause deformation, e.g. plasticity. For plane stress, the material is free to deform out of the plane, and the process zone is correspondingly larger.   [20%]

(b) Use the approximation close to the crack tip that $\sigma_1 = \sigma\sqrt{a/2r}$
Find the location where the yield criterion is met, and define this as the process zone radius r_y.
Circular symmetry of the stress field around the crack top requires that $\sigma_1 \sim \sigma_2$
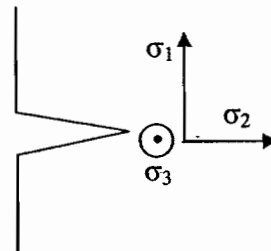
Plane stress, $\sigma_3 = 0$

Using the von Mises criterion
$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 2\sigma_y^2$$
becomes
$$(\sigma_1 - \sigma_1)^2 + (\sigma_1 - 0)^2 + (\sigma_1 - 0)^2 = 2\sigma_y^2$$

$$\sigma_1 = \sigma_y = \sigma\sqrt{\frac{a}{2r_y}} \Rightarrow r_y = \frac{a}{2}\left(\frac{\sigma}{\sigma_y}\right)^2$$

Plane strain $\varepsilon_3 = 0$

Hooke's law gives

$$E\varepsilon_3 = 0 = \sigma_3 - \upsilon\sigma_1 - \upsilon\sigma_2 \Rightarrow \sigma_3 = 2\upsilon\sigma_1$$

von Mises gives $(\sigma_1 - \sigma_1)^2 + 2\sigma_1^2(1 - 2\upsilon)^2 = 2\sigma_y^2 \Rightarrow \sigma_1 = \dfrac{\sigma_y}{1 - 2\upsilon} \Rightarrow r_y = \dfrac{a}{2}\left(\dfrac{(1 - 2\upsilon)\sigma}{\sigma_y}\right)^2$

For metals with $\upsilon \sim 0.3$, this theory predicts that σ_1 for yield with plane strain is increased by a factor ×2.5 compared with plane stress and the process zone reduced by a factor ×0.16. In fact it turns out that yielding around cracks in thick specimens reduces the constraint so that the process zone size is larger than this.   [40%]

(c) A propagating crack has to do work in creating new surface area, and in plastic deformation (or other energy—absorbing process) in the process zone. The energy required to create unit area of crack is called toughness or critical strain energy release rate, and is given the symbol G_{IC}. Increasing this energy increases the toughness. Toughness is strongly influenced by the volume of material which can deform at a crack tip.

In brittle materials (glasses, ceramics, hard metals such as martensite) the process zone is very small. Even if energy—absorbing processes can occur, their overall contribution to the work of fracture is limited by the small size of the plastic zone. Fracture of brittle materials is therefore dominated by the work required to create new surface as bonds break and the crack extends. In ductile materials, the capacity for absorbing energy in the process zone can be very high indeed, as the size of the process zone, and hence the volume of deforming material increases. Thus, a higher strength material will tend to have a larger process zone and so a higher toughness. However broad conclusion can be confounded by changes in the details of the process zone mechanism. For example, heat treatment of steel can increase the strength, but because of the reduced ductility voids can form more easily and the volume of the process zone falls. Hence the amount of energy absorbed falls correspondingly, leading to a reduced toughness. [15%]

(d) (i) Typically polymers are brittle below their glass transition temperature as the increased strength of the cross-linked van de Vaals bonds is offset by reduced ductility and formation of cracks. Introduction of very small rubber particles (in size typically between 0.1 and 10 μm) increases the effective strength at the tip of the crack and the size of the process zone, increasing the toughness. In other words the energy absorbed by the rubber particles increases the toughness.

Rubber-toughened polymers.
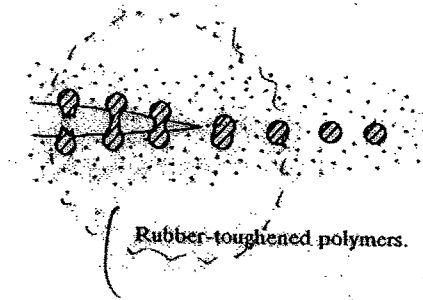
(ii) Liquid mercury causes liquid metal embrittlement (LME), which would be very dangerous to structural components (and mercury has a habit, when it escapes, of getting in all sorts of places). In any case mercury is poisonous. LME is another form of stress corrosion cracking. What is needed is a tensile strain, and a specific combination of liquid metal and solid (host) metal. The symptoms are that host metal which is in contact with a liquid metal suddenly fails by intergranular fracture.

The mechanism depends on the material combination. What happens is that the metal is able to run along inside cracks. It then diffuses a limited distance into the host metal. It reduces the bond strengths at the crack tip, the crack tip process zone size becomes negligibly small (as in brittle fracture), so the fracture toughness falls dramatically. Because it is particularly easy for metal to diffuse along grain boundaries, this type of LME is nearly always intergranular. [25%]

[The qualitative element, deriving a process zone size for plane stress and strain (part b), was not in general well done, while the remaining discussion questions were answered rather better. Need to explain how the constraint in plane strain makes a difference.]

2 (a) Often the problem depends on a wide range of geometric, material and heat transfer properties. For example, welding depends on the heat input, speed, plate thickness and the heat transfer. Dimensional analysis allows the number of parameters to be compressed. This is useful for numerical solutions, where the number of runs is reduced, but also for presentation of the results (for example design charts). Dimensional analysis can highlight the relevant physical phenomena, in the welding example identifying whether 1D or 2D heat flow is necessary. Finally, dimensional analysis can help give suitable ways of representing experimental data, to allow appropriate interpolation or extrapolation to other situations.

(b) $\quad T(t) - T_o = f(\bar{q}, t, a, \lambda)$

NB Clearly need to include time, as T is specified as a function of time.

Assume of the form: $\quad T(t) - T_o = C \, \bar{q}^a \, a^b \, \lambda^c \, t^d$

Units: $\qquad (K) = \left(\dfrac{J}{s\,m^2}\right)^a \left(\dfrac{m^2}{s}\right)^b \left(\dfrac{J}{K\,m\,s}\right)^c (s)^d$

Or in dimensions: $(\theta) = \left(\dfrac{(M\,L^2\,T^{-2})}{T\,L^2}\right)^a \left(\dfrac{L^2}{T}\right)^b \left(\dfrac{(M\,L^2\,T^{-2})}{\theta\,L\,T}\right)^c (T)^d$

Since dimension M only appears in Joules, it is easier to keep the dimensions of Joules together. Considering the units/dimensions in turn:

K (i.e. θ):	$1 = -c$	$c = -1$
J (i.e. $M\,L^2\,T^{-2}$):	$0 = a + c$	$a = 1$
m (i.e. L):	$0 = -2a + 2b - c$	$b = \frac{1}{2}$
s (i.e. T):	$0 = -a - b - c + d$	$d = \frac{1}{2}$

$$T(t) - T_o = C \, \bar{q}^1 \, a^{1/2} \, \lambda^{-1} \, t^{1/2} = C \frac{\bar{q}}{\lambda} \sqrt{a\,t}$$

Or do, perhaps more straightforwardly, using elimination of variables.

Example application: laser processing of metals (e.g. to melt the surface to add alloying additions to a thin surface layer for wear resistance). [45%]

(c) (i) This is an example of so-called static fatigue (actually no cyclic loading is involved). Most 'ordinary' glasses contain sodium as a network modifier (breaks up silicate chains, so reduces softening temperatures and promotes a wide temperature range in which glass can be worked). If the glass is in contact with moisture (particularly in an acid environment), the sodium atoms which terminate the silicate chains can be replaced by hydrogen atoms:

[(silicate chain ending with) -Si-O-Na] + H^+ >> [(chain ending with) -Si-O-H] + Na^+
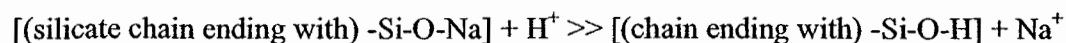
This is associated with shrinkage of the surface layers of the glass. The glass is put into tension and surface cracks form. Cracks grow slowly in the moist environment, and the stress concentrations in the regions of the crack tips mean that the material here is particularly susceptible to corrosion. Once one crack reaches a critical length, fast fracture occurs.

2(c)(ii) Miner's (empirical) rule for high cycle fatigue says that the specimen fails when the proportion of the life time used up by each block adds up to

$$\text{i.e. } \sum_i \frac{N_i}{N_{fi}} = 1 \text{ (Data Book)}$$

N_{fi} is the number of cycles that you would need for failure with the stress range and mean stress of the ith block. Given that it is an empirical rule, we can expect changes in loading pattern to make a difference, e.g. due to changes in initiation mechanisms with differences in load cycle.

Specifically dwell time and waveform do not affect the prediction. For example, Miner's rule will disregard the fact that all the sketched stress variations have different waveforms and dwell times.
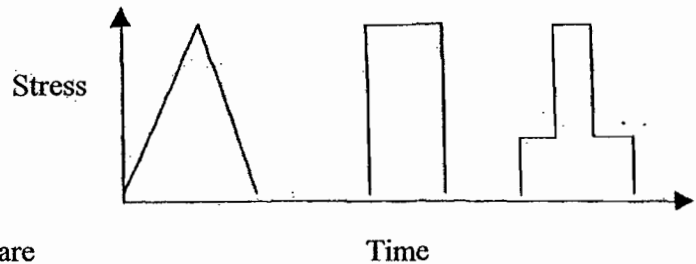
The maximum load and minimum load are the same for all stress pulses, and so all are counted as equivalent, but they may not be:

e.g. fast loading may not have the same effect as slow loading; e.g. if the sample is kept under load, it may creep.

(iii) In uncracked structures cracks nucleate at

    Stress concentrations (e.g. rivets; welds; section changes; inclusions etc.)

   Surfaces (especially at machining marks; corrosion sites etc.)

These cracks grow by fatigue.

Structures containing pre-existing cracks (including most structures welded by traditional methods), skip this nucleation phase, which can be the longest part of the lifetime, particularly for high cycle fatigue (where the stresses are correspondingly lower and nucleation is more difficult). Hence their lifetime can be much shorter.

(iv) During the welding process chromium carbide can form at the grain boundaries, leaving a Cr-depleted zone which isn't protected against corrosion by the normal Cr_2O_3 layer that is formed with stainless steel.   [55%]

[Reasonable attempts to the dimensional analysis (DA) question. In part (a) the relevance of DA both to experimental and numerical/analytical modelling was missed by most. In the main the miscellaneous questions about various aspects of fracture were answered OK.]

3.
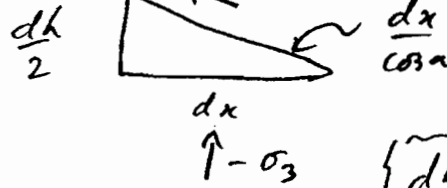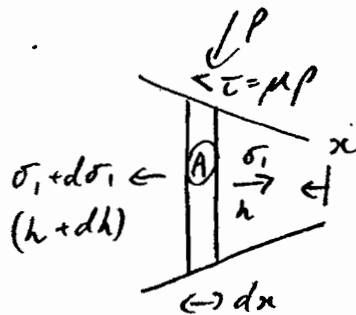
p is pressure $\Rightarrow$ +ve is compression
σ is stress so +ve is tensile

$$\frac{dh}{dx} = 2 \tan \alpha \quad (1)$$

Resolve horizontally on element A $\leftarrow$ (per unit depth)

two surfaces

$$\sigma_1 \, dh + h \, d\sigma_1 + 2p \frac{dx}{\cos\alpha} \cdot \sin\alpha + 2\mu p \frac{dx}{\cos\alpha} \cdot \cos\alpha = 0 \quad (2)$$

resolving

Resolve vertically on triangle (neglecting second order shear mom't) ↑

$$-\sigma_3 \, dx - p \frac{dx}{\cos\alpha} \cdot \cos\alpha + \mu p \frac{dx}{\cos\alpha} \cdot \sin\alpha = 0 \quad (3)$$

1 is tensile, 3 is compressive, 2 has no ε. From flow rule 2, having the intermediate ε, also has the intermediate σ.

Hence Tresca becomes $(\sigma_1 - \sigma_3) = \sigma_y \quad (4)$

Substituting (4) into (3) $\Rightarrow$ $p = \dfrac{(\sigma_y - \sigma_1)}{1 - \mu \tan\alpha} \quad (5)$

Substituting into (2), using (1)

$\boxed{A = \dfrac{1 + \mu/\tan\alpha}{1 - \mu \tan\alpha}}$

$$\Rightarrow \sigma_1 \, dh + h \, d\sigma_1 + p\left(1 + \frac{\mu}{\tan\alpha}\right) dh = 0$$

$$\Rightarrow \int_{h_i}^{h} \frac{dh}{h} = -\int_0^{\sigma_1} \frac{d\sigma_1}{\sigma_1 + A(\sigma_y - \sigma_1)} = \int_0^{\sigma_y} \frac{d\sigma_1}{(1-A)\sigma_1 + A\sigma_1}$$

$$\Rightarrow \ln\left(\frac{h}{h_i}\right) = -\frac{1}{1-A} \ln\left(\frac{(1-A)\sigma_1 + A\sigma_y}{A\sigma_y}\right)$$

Rearrange to give $\sigma_1 = \dfrac{A}{A-1}\left\{1 - \left(\dfrac{h}{h_i}\right)^{A-1}\right\} \cdot \sigma_y$ [60%]

[Not easy to get to the right answer, but good marks awarded when all the elements were in place.]

3 (b) $h_o/h_i = \frac{2}{3}$

$\tan\alpha = \dfrac{(h_i - h_o)/2}{w} = 0.25$

$A = \dfrac{1 + 0.7 \times 4}{1 - 0.7 \times 4} = 4.606$

$\sigma_{1,0} = \dfrac{4.606}{3.606}\left\{1 - \left(\frac{2}{3}\right)^{3.606}\right\} = 0.98\,\sigma_y$

This is nearly sufficient for the sheet to fail
in tension at the die exit, which would lead
to necking there instead of the sheet being
pulled through the die.

We need to reduce the outlet tension.

The equation shows that reducing the yield stress
won't reduce the exit stress, as a proportion of
the yield stress. Intuitively reducing μ, eg by using
a better lubricant, will help.

Check: $\mu = 0.5 \Rightarrow A = 4 \Rightarrow \sigma_1 = 0.96$.   Better

But eqn tells us we could reduce A and hence $\sigma_{1,0}$
by increasing α.

eg $\tan\alpha = 0.3 \Rightarrow A = 4.2$.   Better.

It is not clear whether increasing the back tension will
help without further analysis: my analysis suggests that, for $A > 1$,
adding back tension increases the exit stress. An exercise for the reader!
More radical solutions include reducing the reduction (but
then need more passes), applying compression at entry to make
the process an extrusion, or switching to rolling.

[People tended not to relate their suggestion to the
equation derived, and mentioned back tension without thinking.]

4. (a)   Treat as a single infinite material, with the centre of the billet at $x=0$, and the interface between the dies and billet at $x=\pm l$. The temperature is first made dimensionless, by scaling with the initial conditions: $[T(x,t) - 20]/[900 - 20]$.

In terms of this dimensionless temperature, the given boundary conditions are as in Problem (1) below.

By superposition, Problem (1) $=$ Problem (2) $+$ Problem (3).

Problem (1)
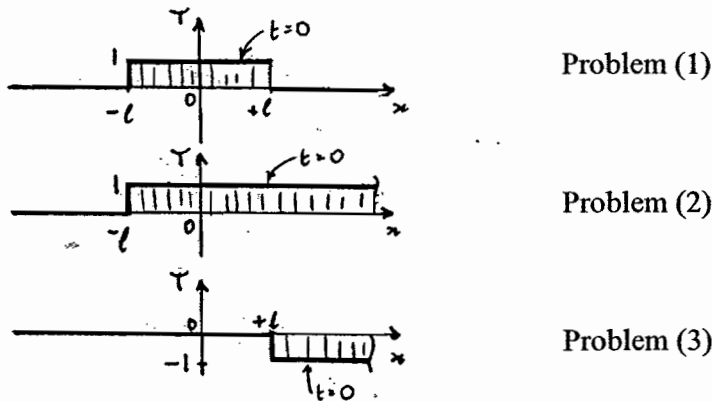
Problem (2)

Problem (3)

Solutions to Problems (2) and (3) can be deduced from the solution for Problem (4) below, by shifting the origin by a distance $\pm l$, and multiplying by (-1) in the latter case.
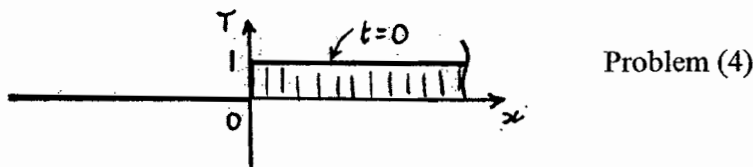
Problem (4)

The solution to Problem (4) may be scaled from the solution in the question – first by halving it, and then by a vertical shift of $+\frac{1}{2}$ (as the range is 0 to 1, instead of -1 to 1). Hence the solution will be:

$$T(x,t) = \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{2\sqrt{at}}\right) + \frac{1}{2}$$

Change variable $x \rightarrow x+l$ to get solution to problem (2):

$$T(x,t) = \frac{1}{2}\,\mathrm{erf}\left(\frac{x+l}{2\sqrt{at}}\right) + \frac{1}{2}$$

Change variable $x \rightarrow x-l$ to get solution to problem (3), noting that T=-1 for $x > l$ at t=0:

$$T(x,t) = -\frac{1}{2}\,\mathrm{erf}\left(\frac{x-l}{2\sqrt{at}}\right) - \frac{1}{2}$$

Hence solution to problem (1) = solution to problem (2) + solution to problem (3):

$$\frac{T(x,t)-20}{900-20} = \frac{1}{2}erf\left(\frac{x+l}{2\sqrt{at}}\right) - \frac{1}{2}erf\left(\frac{x-l}{2\sqrt{at}}\right)$$

[65%]

(b) Solution for centre ($x = 0$) (equation 1):

$$\frac{T(0,t) - 20}{900 - 20} = \tfrac{1}{2} erf\left(\frac{l}{2\sqrt{at}}\right) - \tfrac{1}{2} erf\left(\frac{-l}{2\sqrt{at}}\right) = erf\left(\frac{l}{2\sqrt{at}}\right)$$

and solution for interface ($x = l$) (equation 2):

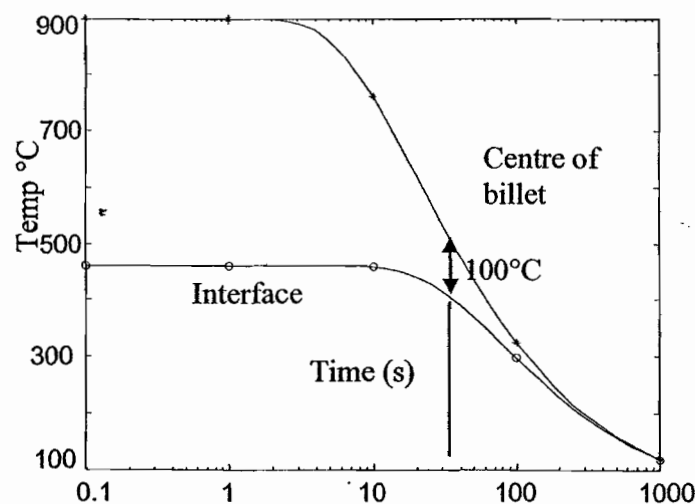$$\frac{T(l,t) - 20}{900 - 20} = \tfrac{1}{2} erf\left(\frac{2l}{2\sqrt{at}}\right) - \tfrac{1}{2} erf\left(\frac{l-l}{2\sqrt{at}}\right) = \tfrac{1}{2} erf\left(\frac{l}{\sqrt{at}}\right)$$

Here $l = 0.02$m and $a = 10^{-5}$ m^{2}s^{-1}:

i.e. $\left(\dfrac{l}{2\sqrt{at}}\right) = \dfrac{0.02}{2\sqrt{10^{-5}t}} = \dfrac{1}{\sqrt{0.1t}}$ (eqn 3) & $\left(\dfrac{l}{\sqrt{at}}\right) = \dfrac{0.02}{\sqrt{10^{-5}t}} = \dfrac{2}{\sqrt{0.1t}}$ (eqn 4)

Get the error function values from the Materials Databook.

t (s)	Centre			Interface		
	$\left(\dfrac{l}{2\sqrt{at}}\right)$ eqn 3	$erf\left(\dfrac{l}{2\sqrt{at}}\right)$	$T(0, t)$ (eqn 1)	$\left(\dfrac{l}{\sqrt{at}}\right)$ eqn 4	$erf\left(\dfrac{l}{\sqrt{at}}\right)$	$T(0, t)$ eqn 2
0.1	10	1.00	900	20	1	460
1	3.16	1.00	900	6.4	1	460
10	1	0.84	760	2	0.995	458
100	0.316	0.35	320	0.64	0.63	297
1000	0.1	.011	120	0.2	0.22	118

From the graph (more detailed than required), the centre and surface of the billet differ in temperature by 100°C after approximately 35 seconds.   [35%]

[A very bimodal distribution, with this question either answered by the brave or the less able. Part (a) was rather found to be rather straightforward by many. Part (b) had the tendency to take rather a long time, though there were some correct solutions. Good marks were awarded for (b) for deriving appropriate equations and having a stab at the sketches.]