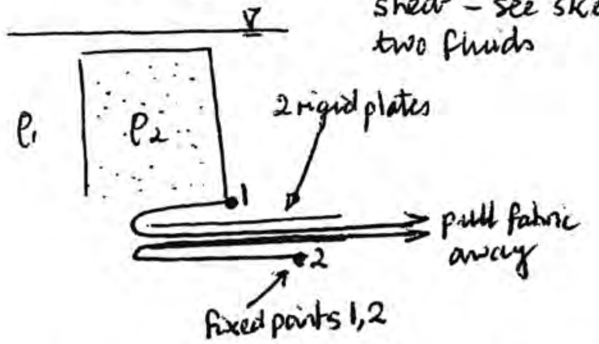
# 4A10 CRIB

1.

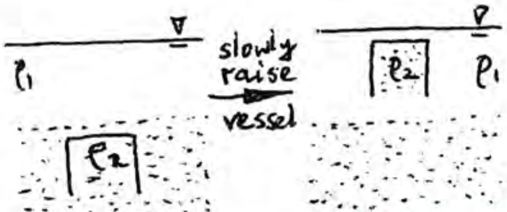
(a) Oil/water or oil-oil and clear plexiglass vessel for flow visualisation. Of possible liquid combinations, select two with large surface tension – this can be reduced in final set up by a surfactant.

– possible approach : with vessel in position/immersed in liquid 1, introduce, say, via tube in top liquid 2 slowly / potentially with 'false' lid that is displaced downwards. [Or even carefully invert when filled with liquid 2 – this will be subject to disturbances]

– better approach : use a 'shear free' lid – two plates wrapped in fabric to prevent shear – see sketch – fabric is 'unrolled' along interface between the two fluids

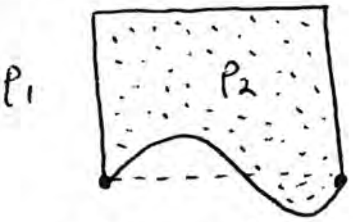



– credit given for other sensible/plausible approaches, eg

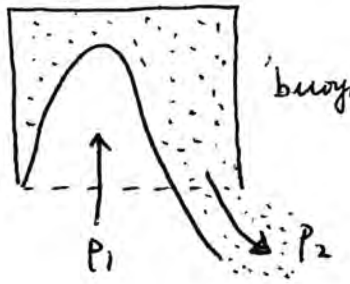



[3 marks]

(b) Initial/base state is 'top heavy', the stabilising effect of surface tension dominates the destabilising effect of buoyancy force. Surfactant would reduce surface tension – disturbances present in base state could grow to produce standing waves on interface (eg mode shown left) if growth saturated or continued growth could result in flushing of liquid 2 from the vessel (right)




surface tension dominated



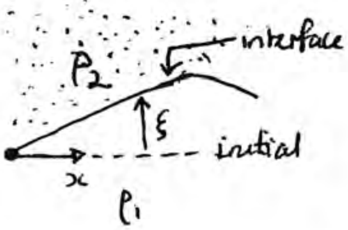

buoyancy dominated

[3 marks]

(c) Change in energy of system $\Delta E$ comprised of a gravitational potential energy and surface energy (surface tension) component

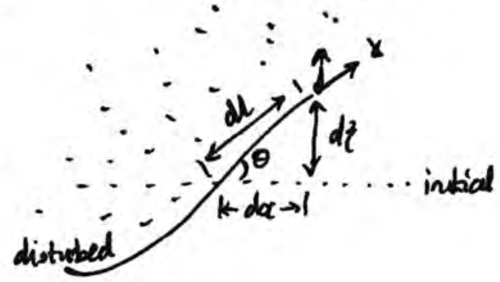
$$\Delta E = \Delta GPE + \Delta X \qquad X \text{ surface tension.}$$

For an element width $dx$ with unit length (into page) of vessel




$$\Delta GPE = \left(\frac{1}{2}\,dx\,\xi^2 \times 1\right) \times g(\rho_1 - \rho_2)$$

1 (c) Change in energy of system comprised of gravitational potential energy & Surface energy component $\Delta E = \Delta GPE + \Delta \gamma$




surface energy : initially $\gamma \underbrace{dx \times 1}_{\text{element of area}}$

: when disturbed $\gamma \, dl \times 1$

Now $\sin\theta = \dfrac{d\zeta}{dx} \approx \theta$ for small $\theta$ and

$$dl^2 = dx^2 + d\zeta^2$$

$$= dx^2\left(1 + (d\zeta/dx)^2\right) \Rightarrow dl \approx dx\left(1 + \tfrac{1}{2}(d\zeta/dx)^2\right)$$

retaining leading order/ neglect terms of $O(d\zeta/dx)^4$ and higher

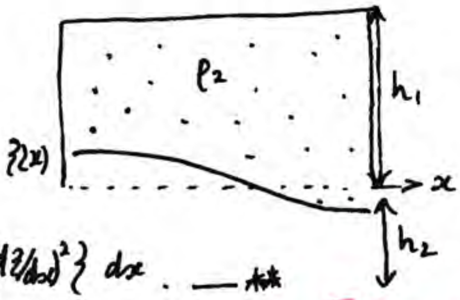
So $\Delta\gamma = \gamma(dl - dx) = \dfrac{\gamma}{2}\left(\dfrac{d\zeta}{dx}\right)^2 dx$

Disturbed

$$GPE(x) = \left( \int_{-h_2}^{\zeta(x)} \rho_1 g z \, dz + \int_{\zeta(x)}^{h_1} \rho_2 g z \, dz \right) dx = \left( \frac{\rho_1 g}{2}(\zeta^2 - h_2^2) + \frac{\rho_2 g}{2}(h_1^2 - \zeta^2) \right) dx$$

and base state

$$GPE(x) = \left( \int_{-h_2}^{0} \rho_1 g z \, dz + \int_{0}^{h_1} \rho_2 g z \, dz \right) dx$$




$\Rightarrow \Delta GPE = (\rho_1 - \rho_2)\dfrac{g}{2}\zeta^2 \, dx$

& so total energy change (disturbed − undisturbed) $= \underline{\displaystyle\int_0^d \left\{ \frac{g}{2}(\rho_1 - \rho_2)\zeta^2 + \gamma/2 \, (d\zeta/dx)^2 \right\} dx}$ . — **

[8 marks]

(d) Normal mode disturbances of form $\zeta = a \sin(2n\pi x/d)$ give $\begin{cases} \zeta = 0 \text{ on } x = 0 \\ \zeta = 0 \text{ on } x = d \end{cases}$ satisfying the no slip. First mode* to go unstable will be $n=1$

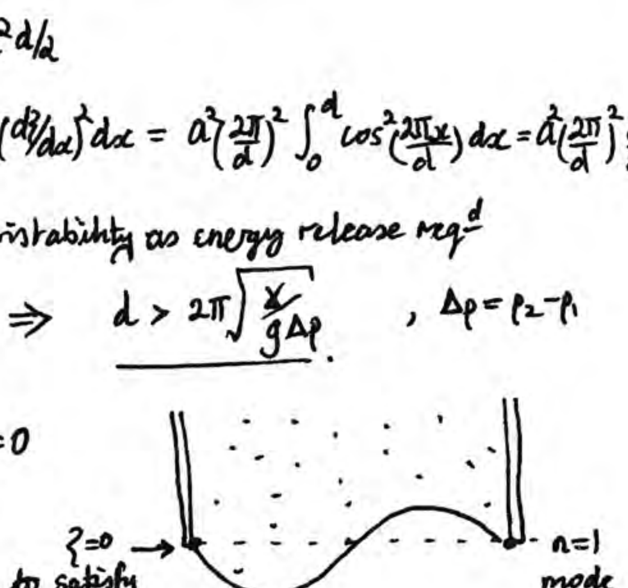
Thus $\displaystyle\int_0^d \zeta^2 \, dx = a^2 \int_0^d \sin^2\left(\frac{2\pi x}{d}\right) dx = a^2 d/2$

$d\zeta/dx = a \cdot \dfrac{2\pi}{d} \cos\left(\dfrac{2\pi x}{d}\right)$ so that $\displaystyle\int_0^d (d\zeta/dx)^2 dx = a^2\left(\frac{2\pi}{d}\right)^2 \int_0^d \cos^2\left(\frac{2\pi x}{d}\right) dx = a^2\left(\frac{2\pi}{d}\right)^2 \frac{d}{2}$

Sub in ** noting that $\Delta GPE < 0$ for instability as energy release req$^d$

$$-\left(\frac{a^2 d}{2}\right)\frac{g}{2}\Delta\rho + \frac{\gamma}{2}\left(\frac{a^2 d}{2}\right)\left(\frac{2\pi}{d}\right)^2 < 0 \quad \Rightarrow \quad \underline{d > 2\pi\sqrt{\frac{\gamma}{g\Delta\rho}}} \; , \; \Delta\rho = \rho_2 - \rho_1$$

* that conserves volume given $\displaystyle\int_0^d \zeta(x) \, dx = 0$




[6 marks]

Q1 comments

This question concerned the stability of an interface with surface tension that separates an upper region of dense liquid from a lower region of less dense liquid. The question began by asking how this gravitationally unstable system could be set up in the laboratory and using which liquids, and proceeded to enquire as to the form of instability expected when perturbed. This was generally answered well. The question then required the derivation of an integral expression (given) relating the energy of the system in the undisturbed and disturbed states. The surface energy component was almost universally derived well - the gravitational potential energy component was not, with candidates proposing expressions largely in the absence of any clear reasoning or working backwards from the expression given. The final part of the question was to infer a constraint on the geometry of the system that would ensure stability. A number of students successfully tackled this, introducing a form of disturbance that would satisfy the no-slip boundary conditions and conserve volume. Others identified the correct form of disturbance but made errors in integration.

2 (a)

(i) Given $\frac{\partial u}{\partial t} - \sin u = \frac{1}{R}\frac{\partial^2 u}{\partial z^2}$ and $u=0$ on $z=0,\pi$

For base flow $\partial/\partial t = 0$. By inspection $U = u = 0$ is base solution satisfying the governing equation & b.c.'s.

Perturb so that $u = 0 + u'$ $\Rightarrow$ $\frac{\partial u'}{\partial t} - \sin u' = \frac{1}{R}\frac{\partial^2 u'}{\partial z^2}$

For infinitesimally small $u'$, $\sin u' = u'$, so linearised system is

$$\underline{\frac{\partial u'}{\partial t} - u' = \frac{1}{R}\frac{\partial^2 u'}{\partial z^2}}, \quad u'=0 \text{ on } z=0,\pi. \qquad \text{—(1)}$$

[2 marks]

(ii) There are multiple routes to solution. I will express perturbation as Fourier series (noting the 2nd spatial derivative in (1))

$$u'(z,t) = \sum_{n=1}^{\infty} A_n(t) \sin nz \qquad \text{—(2)}$$

and decomposition will reduce to o.d.e in time & I'll then look for sol's of form $A_n(t) = A_0 e^{st}$. Or, one could instead begin with $u' = \hat{u}(z) e^{st}$. Clearly, all possible waveforms of disturbance can be expressed by (2).

Sub. for (2) into (1) gives $\quad \sum_{n=1}^{\infty} \sin nz \left( \frac{dA_n}{dt} - A_n + \frac{n^2}{R}A_n \right) = 0$

Could now $\times \sin mz$ and integrate between 0 and $\pi$ w.r.t. $z$ [& use fact $\int_0^\pi \sin nz \sin mz \, dz$ is zero unless $n=m$. Either way, we have

$$\frac{dA_n}{dt} - A_n + \frac{n^2}{R}A_n = 0$$

With $A_n(t) = A_0 e^{st}$ we obtain $\quad s - 1 + \frac{n^2}{R} = 0 \quad \Rightarrow \quad \underline{s = 1 - n^2/R}$ is the growth rate.

[8 marks]

(iii). Stability requires $s < 0$ & thus for $\frac{n^2}{R} > 1$, i.e. for $R < n^2$

Minimum value of $n$ is 1, giving $R < 1$. We are given in the question $R > 0$. $\therefore$ Stability for $\underline{0 < R < 1}$.

If the sign of the trigonometric term is changed then $\underline{s = -1 - n^2/R}$ & the system is stable

[2 marks]

2(b)

(i) Setting $\partial/\partial t = 0$, it is clear that $U = u = 0$ is base solution.

We are instructed to seek normal mode sol$^{\underline{n}}$ $u = ce^{i(kx - \omega t)}$

Thus $\dfrac{\partial u}{\partial t} + V\dfrac{\partial u}{\partial x} = \sigma u + \nu\dfrac{\partial^2 u}{\partial x^2}$

reduces to

$$-i\omega + Vik = \sigma + \nu(ik)^2 \qquad — (*)$$

So that the dispersion relationship $\mathcal{D}(k,\omega)=0 \Rightarrow \underline{\mathcal{D}(k,\omega) = -i(\omega - kV) - \sigma + \nu k^2}$

[6 marks]

(ii) aside $\begin{cases} e^{i(kx-\omega t)} \equiv e^{ikx + st} \quad \text{giving } s = -i\omega \\ \text{Now } \omega = \omega_R + i\omega_I \quad \Rightarrow \quad s = -i\omega_R + \omega_I \text{. Hence } Re\{s\} = \omega_I \\ \text{i.e. require } \underline{\omega_I < 0 \text{ for stability}} \quad (\text{giving } Re\{s\} < 0\} \end{cases}$

From (*) $\quad s = -i\omega = \sigma - \nu k^2 - Vik$

$\quad Real\{s\} = \sigma - \nu k^2 < 0 \quad$ for stability

so $\underline{\sigma < 0}$. $\quad$ gives $Real\{s\} < 0 \ \forall k > 0$

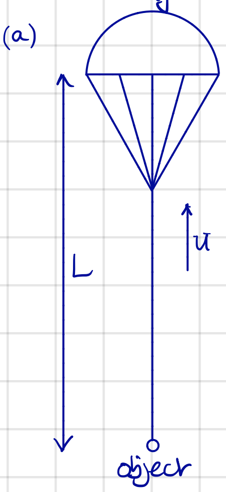
[2 marks]

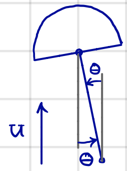
Q2 comments

This question was designed to assess the candidates' essential understanding and ability to execute and interpret the results of a linear stability analysis. The question consisted of two parts. In the first, the students were required to linearise a form of the diffusion equation which was modified to include a trigonometric sin term. Almost no-one spotted that the base solution was the trivial solution U=0 and instead introduced perturbations about an unknown base state U – this then led them to considerably overcomplicate the solution and stifle their interpretation. Most surprisingly, the sin term caused a number of unanticipated problems as candidates either did not, or did not immediately, reduce sin to its argument on linearising. Very few reasoned why the approach they took could be regarded as treating all possible small amplitude waveforms of disturbance. Given so very few attempted to make this reasoning, I question whether this part of the question was missed. For the second part of the question, candidates were given an already linearised system and required to establish and then interpret the dispersion relationship. This was generally answered very well by all.

3. Change to a frame of reference moving vertically with the parachute.

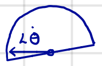
(a)



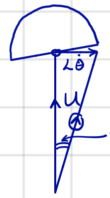




Define $\theta$. When $\dot{\theta}$ is positive, the parachute moves horizontally left.

frame stationary horizontally →




frame moving horizontally with parachute →




angle of attack increased by $\tan^{-1}\left(\frac{L\dot{\theta}}{U}\right) \approx \frac{L\dot{\theta}}{U}$ because $U \gg L\dot{\theta}$

Make the quasi-steady assumption that the fluid forces on the parachute during motion can be modelled as those from a steady flow in the same instantaneous configuration.

Before accounting for horizontal motion of the parachute, the angle of attack of the parachute, relative to the oncoming flow, is simply $\theta$. After accounting for the horizontal motion, the apparent angle of attack is:

$$\alpha = \frac{L\dot{\theta}}{U} + \theta$$

Most students answered this well. It required a diagram & some justification.

[4 marks]

(b) Consider moments around the object and write an expression for $\ddot{\theta}$:

b and c have the same sign as k.

$$mL^2\ddot{\theta} = Lk\alpha = Lk\left\{\frac{L\dot{\theta}}{U} + \theta\right\} \Rightarrow \ddot{\theta} - \overbrace{\frac{k}{mU}}^{b}\dot{\theta} - \overbrace{\frac{k}{mL}}^{c}\theta = 0$$

Most students answered this well. Small algebra errors were common.

[4 marks]

(c) Although the behaviour can be described by inspection, let us propose an ansatz $\theta = \theta_0 e^{st}$:

$$\Rightarrow s^2\theta_0 - sb\theta_0 - c\theta_0 = 0 \quad \Rightarrow \theta_0 = 0 \text{ or} \ldots$$
$$\Rightarrow 2s = b \pm (b^2 + 4c)^{1/2}$$

If $k > 0$ then $b > 0$ and $c > 0$ so $s$ has at least one positive real root. This is torsional divergence. Within this model, the fluid force is always in the direction of the parachute's horizontal motion. This fluid force model will break down at large angles $\alpha$.

If $k < 0$ then $b < 0$ and $c < 0$. If the term in the square root is negative, which occurs when $k/mU < 4$ then this corresponds to a damped oscillating motion. If the term in the square root is positive then there are two negative real solutions, which corresponds to damped non-oscillating solutions.

In summary, a parachute must be designed such that $k < 0$. Limitations are that no aerodynamic or mechanical damping has been included; the model is only valid for small angles;

Many students examined the sign of the damping term in the equation of motion and stated that the parachute would be unstable to gallop - only a few said that it would be unstable to torsional divergence. The candidates who tried the ansatz $\theta = \theta_0 e^{st}$ gave more detailed and better answers.

[8 marks]

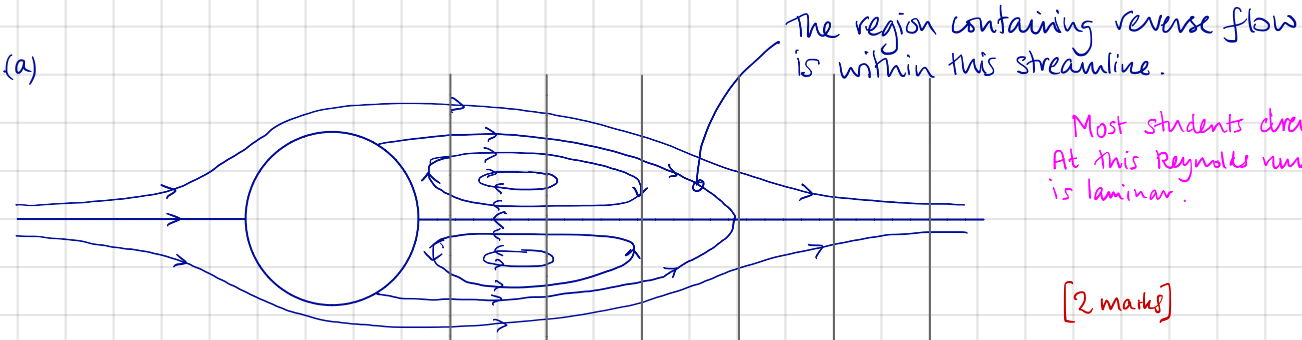
(d) Let us model the hemi-spherical parachute as the D-shaped cross section in the notes and lectures. For speeds, $v$, of this cross section less than the free stream $U$, we recall that the aerodynamic force was in the direction of the motion of the cross section. This corresponds to $k > 0$, which is unstable.

Round parachutes are indeed not used because they are unstable, unless they have a hole in the top.

Around half of the candidates answered this well. Some gave lengthy answers about vortex shedding, which was not relevant to this question.

[4 marks]

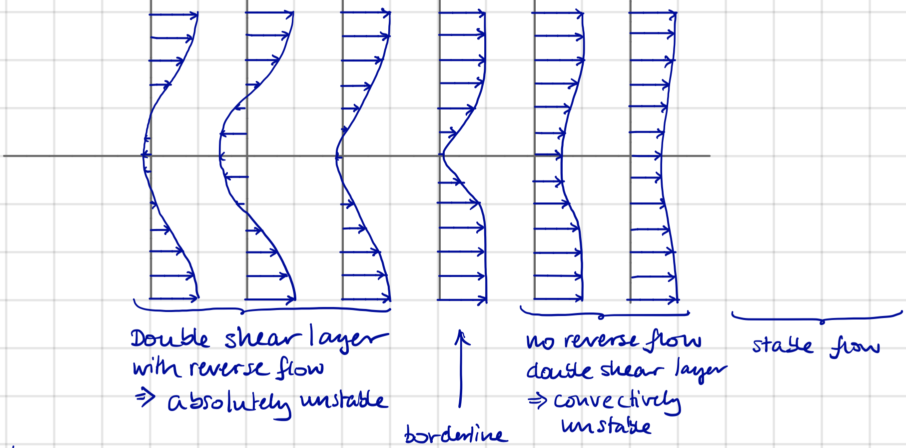
4 (a)




Most students drew this well. At this Reynolds number the flow is laminar.

[2 marks]

(b)



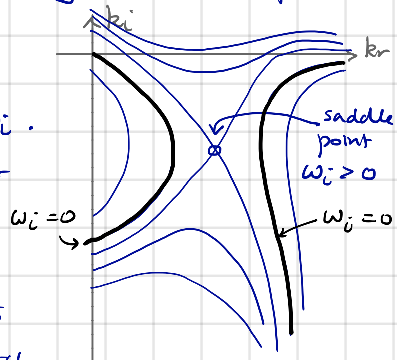


This section was answered well by many students.

The flow is absolutely unstable to sinuous oscillations in a large region behind the cylinder. In this region, sinuous oscillations with zero group velocity grow until they saturate (i.e. reach maximum amplitude). This is known as global instability. This region of the flow oscillates at a fixed Strouhal number. This oscillation is insensitive to external forcing (unless the forcing is very strong). The rest of the flow responds to this forcing. For this flow, the sinuous oscillation saturates nonlinearly into vortex shedding. A pattern of shed vortices forms in the downstream flow.

[6 marks]

(c) Consider the flow in the absolutely unstable region, where there is reverse flow. We can model this as a 2D wake with infinitely-thin shear layers. The dispersion relation determines $\omega(k)$. In this region, contours of $\omega_i(k)$ are sketched on the right and the saddle point, which is the wave with zero group velocity, has positive growth rate, $\omega_i$. The thick black line shows the spatial stability analysis, for which $\omega_i = 0$. The saddle point is at higher $\omega_i$ than this. The spatial analysis has no peak (i.e. no minimum $k_i$), which corresponds to the max. spatial growth rate. This is because the spatial analysis has no meaning in an absolutely unstable flow; the forced response is drowned out by the intrinsic response of the wave with zero group velocity.

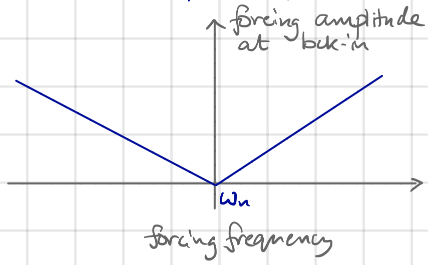



[6 marks]

This was answered well by many students. The main point is the last one.

(d) The flow oscillates at Strouhal $\simeq 0.2$. This corresponds to cyclic frequency $f_n = St\, U/D$ and angular frequency $\omega_n = 2\pi f_n \simeq 2\pi \times 0.2\, U/D$. If the forcing frequency, $\omega$, is close to $\omega_n$ then, when the forcing amplitude exceeds some threshold, the flow's oscillation frequency will lock into the forcing frequency.

The forcing amplitude at lock-in varies with the forcing frequency according to this graph →




[6 marks]

Only half the students answered this well. The question asks what happens as the oscillation amplitude increases. Some described what happens as the flow speed increases.