EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 23 April 2024 9.30 to 11.10

Module 4A10

FLOW INSTABILITY

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4A10 data sheet (two pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 Consider the initial situation depicted in Fig.1. An upturned vessel of uniform width d is immersed in a large body of quiescent liquid of uniform density ρ_1. The length of the vessel (into the page, Fig.1) is much greater than d. The vessel is filled with liquid of uniform density ρ_2 ($> \rho_1$). There is surface tension γ on the interface between the two liquids. The undisturbed interface may be assumed to be horizontal. Let ξ denote the departure of the interface from this initial position, at $z = \xi(x,t) = 0$, when it is subjected to small amplitude disturbances travelling in the x-direction. You may assume throughout that disturbances are solely in the x-direction.

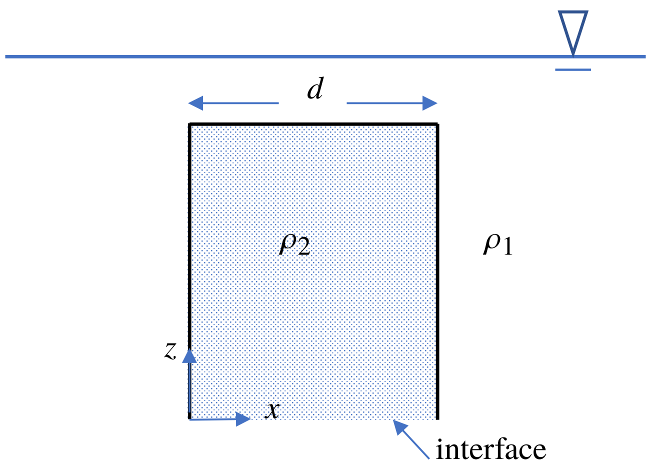


Fig. 1

(a) From a practical perspective, describe how you would set up this initial situation in a laboratory experiment whilst minimising disturbances that could destabilise the system. Name two liquids that would be suitable. [15%]

(b) Describe with physical reasoning what would typically be observed if a surfactant were gently injected on the interface from below. Include sketches to aid your description. [15%]

(c) Suppose the interface is perturbed from the initial horizontal state by a very small amplitude disturbance. With g denoting the acceleration due to gravity, show that the *change* in energy per unit length of vessel (into page) between the initial and perturbed states may be expressed as

$$\int_0^d \left\{ \frac{g}{2}(\rho_1 - \rho_2)\xi^2 + \frac{\gamma}{2}\left(\frac{d\xi}{dx}\right)^2 \right\} dx$$

[40%]

 (cont.

Hint: Consider the surface energy of an elemental area of the interface in undisturbed and disturbed states.

(d) For a small amplitude disturbance $\xi(x)$ that satisfies continuity and the no-slip condition on the walls of the vessel, hence, or otherwise, show that instability requires

$$d > 2\pi \left(\frac{\gamma}{g\Delta\rho} \right)^{1/2}$$

where $\Delta\rho = \rho_2 - \rho_1$. [30%]

2 (a) A model of the one-dimensional flow of fluid with velocity $u(z, t)$ along a channel between the parallel walls $z = 0$ and $z = \pi$ is described by the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} - \sin u = \frac{1}{R}\frac{\partial^2 u}{\partial z^2}$$

where R (> 0) is a Reynolds number. The boundary conditions are $u = 0$ at $z = 0, \pi$.

(i) For infinitesimal perturbations $u'(z, t)$ about the steady base flow, establish the linearised form of the governing equation and boundary conditions that control these perturbations. [10%]

(ii) Solve this linearised problem in order to assess the stability of the model. [40%]

(iii) Deduce the range of R for which the flow is stable. Comment on the implications for stability of a change in the sign of the trigonometric term in the model. [10%]

Credit will be given for brief supporting statements that convey the logic behind your solution approach and clarify how you have analysed the stability with respect to all possible perturbations.

(b) Now consider the linear model equation

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = \sigma u + \nu\frac{\partial^2 u}{\partial x^2}$$

for $V > 0$, $\nu > 0$ and real σ.

(i) Show that the dispersion relation $\mathcal{D}(k, \omega) = 0$ of normal modes with $u \propto e^{i(kx - \omega t)}$ is specified by

$$\mathcal{D}(k, \omega) = -i(\omega - kV) - \sigma + \nu k^2$$

[30%]

where t is time, i the imaginary unit, ω the frequency and k (> 0) the wavenumber.

(ii) Deduce that the null solution is stable if $\sigma < 0$. [10%]

3 A parachute (see Fig.2) is tethered a distance L behind a heavy object falling vertically at speed U. The tether rope is always straight but can rotate around the object, making angle θ with the vertical direction. The angle of attack of the parachute, α, is defined to be zero when $\theta = 0$. The horizontal force on the parachute is given by $F = k\alpha$, where k is determined by the shape of the parachute. You may assume that the horizontal speed of the parachute is always much less than U.

(a) Making the quasi-steady assumption, show that the apparent angle of attack of the parachute is

$$\alpha = \frac{L\dot{\theta}}{U} + \theta$$

[20%]

(b) The mass of the parachute, combined with the added mass of air around it, is m. Stating your assumptions, derive a second order ordinary differential equation for θ in terms of m, L, U and k. [20%]

(c) Describe the motion of the parachute and any limitations of the model when

(i) $k > 0$ [20%]

(ii) $k < 0$ [20%]

(d) Explain why a parachute shaped like a hemi-spherical shell would be dangerous. [20%]

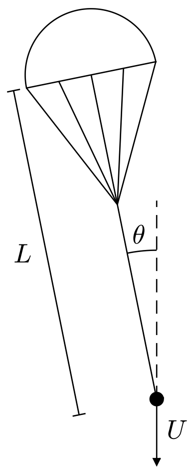


Fig. 2

4 (a) Sketch the streamlines of 2D flow behind a cylinder at Reynolds number 100, indicating the region containing reverse flow. [10%]

(b) By considering the streamwise velocity profile at a few locations behind the cylinder, identify likely regions of absolute instability, convective instability, and stability. Sketch the absolute growth rate as a function of distance behind the cylinder and describe the motion of the flow behind the cylinder. With reference to your previous answers, explain why the flow behaves this way. [30%]

(c) The cylinder is vibrated in the cross-stream direction with very small amplitude at angular frequency ω. With the aid of sketches of permitted $\omega(k)$ in the complex k-plane, where k is the complex wavenumber of small disturbances, or otherwise, explain why the forced response cannot be calculated with a spatial stability analysis. [30%]

(d) Describe the flow's behaviour as the oscillation amplitude, a, increases, explaining how this depends on the free stream velocity, U, the diameter of the cylinder, D, and the forcing angular frequency, ω. [30%]

END OF PAPER

Module 4A10 Data Card

EQUATIONS OF MOTION

For an incompressible isothermal viscous fluid:

Continuity $\nabla \cdot \boldsymbol{u} = 0$

Navier Stokes $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \mu \nabla^2 \boldsymbol{u}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

IRROTATIONAL FLOW $\nabla \times \boldsymbol{u} = 0$

velocity potential ϕ,

$$\boldsymbol{u} = \nabla \phi \text{ and } \nabla^2 \phi = 0$$

Bernoulli's equation

for inviscid flow $\dfrac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + gz + \dfrac{\partial \phi}{\partial t} =$ constant throughout flow field.

KINEMATIC CONDITION AT A MATERIAL INTERFACE

A surface $z = \eta(x,y,t)$ moves with fluid of velocity $\boldsymbol{u} = (u,v,w)$ if

$$w = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta \quad \text{on } z = \eta(x,t).$$

For η small and $\boldsymbol{u}$ linearly disturbed from $(U,0,0)$

$$w = \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} \quad \text{on } z = 0.$$

SURFACE TENSION σ AT A LIQUID-AIR INTERFACE

Potential energy

The potential energy of a surface of area A is σA.

Pressure difference

The difference in pressure Δp across a liquid-air surface with principal radii of curvature R_1 and R_2 is

$$\Delta p = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right).$$

For a surface which is almost a circular cylinder with axis in the x-direction, $r = a + \eta(x,\theta,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = \frac{\sigma}{a} + \sigma \left(-\frac{\eta}{a^2} - \frac{\partial^2 \eta}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 \eta}{\partial \theta^2} \right),$$

where Δp is the difference between the internal and the external surface pressure.

For a surface which is almost plane with $z = \eta(x,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = -\sigma \frac{\partial^2 \eta}{\partial x^2}$$

where Δp is the difference between pressure at $z = \eta^+$ and $z = \eta^-$.

ROTATING FLOW

In steady flows with circular streamlines in which the fluid velocity and pressure are functions of radius r only:

Rayleigh's criterion

The flow is unstable / stable to inviscid axisymmetric disturbances if Γ^2 decreases / increases with r.

$\Gamma = 2\pi r V(r)$ is the circulation around a circle of radius r.

Navier Stokes equation simplifies to

$$0 = \mu \left(\frac{d^2 V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2} \right)$$

$$-\rho \frac{V^2}{r} = -\frac{dp}{dr}.$$

Module 4A10 Data Card

STABILITY OF PARALLEL SHEAR FLOW

Rayleigh's inflexion point theorem

A parallel shear flow with profile $U(z)$ is only unstable to inviscid perturbations if

$$\frac{d^2U}{dz^2} = 0 \quad \text{for some } z.$$

CONVECTIVE FLOW

The Boussinesq approximation leads to

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho_0}\nabla p + (1 - \alpha(T - T_0))\boldsymbol{g} + \upsilon \nabla^2 \boldsymbol{u}$$

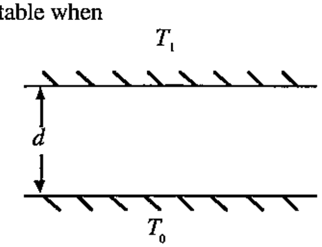
and

$$\frac{DT}{Dt} = \kappa \nabla^2 T$$

Rayleigh-Bénard convection

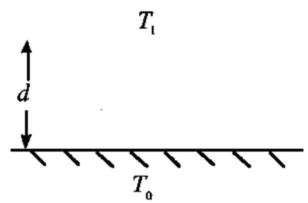
A fluid between two **rigid** plates is unstable when

$$\text{Ra} \geq 1708$$



A liquid with a **free** upper surface is unstable when

$$\frac{\text{Ra}}{\text{Ra}_c} + \frac{\text{Ma}}{\text{Ma}_c} \geq 1$$



where

$$\text{Ra} = \frac{g\alpha(T_0 - T_1)d^3}{\upsilon\kappa}, \quad \text{Ma} = \frac{\chi(T_0 - T_1)d}{\rho\upsilon\kappa} \quad \text{with } \chi = -\frac{d\sigma}{dT}$$

$\text{Ra}_c \approx 670 \qquad \text{Ma}_c \approx 80.$

USEFUL MATHEMATICAL FORMULA

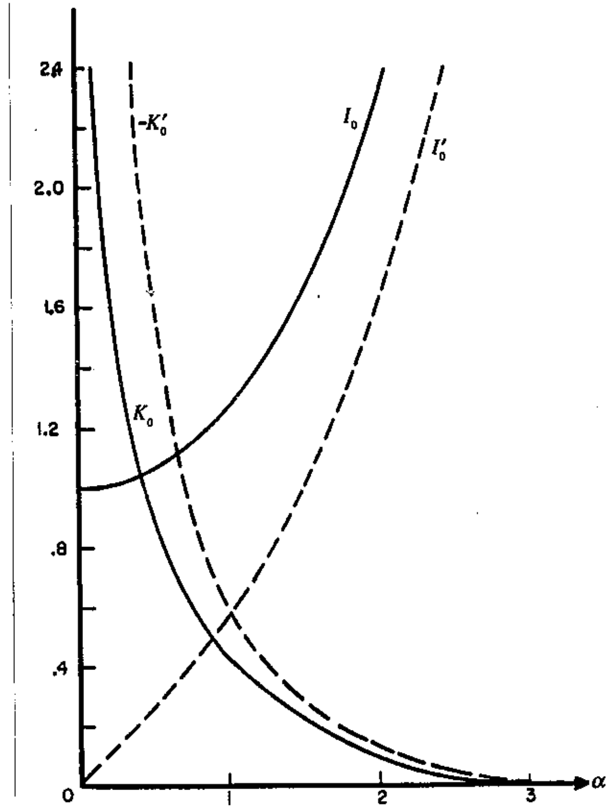
Modified Bessel equation

$I_0(kr)$ and $K_0(kr)$ are two independent solutions of

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - k^2 f = 0.$$

$I_0(kr)$ is finite at $r = 0$ and tends to infinity as $r \to \infty$,

$K_0(kr)$ is infinite at $r = 0$ and tends to zero as $r \to \infty$.



$I_0(\alpha)$, $K_0(\alpha)$, $I_0'(\alpha)$, $K_0'(\alpha)$
where ′ denotes a derivative