ENGINEERING TRIPOS PART IIB 2023
MODULE 4A10

Detailed comments

Question 1

Linear stability of a shear flow

This question, attempted by the majority, required the manipulation and interpretation of a dispersion relationship governing the linear stability of a shear flow with surface tension. Whilst not at all straightforward, the vast majority of attempts were generally sound, with a number being excellent.

Question 2

Flow instability

This question, attempted by the majority, challenged the students on a broad range of material across the course, spanning energy arguments for centrifugal instability and density stratified shear flows together with aspects of capillary jet instability. Unlike question 1, this question required knowledge of some bookwork and although generally tackled well, numerous solutions gave solutions that were somewhat sloppy or incomplete, missing key aspects of the rigour required for the derivations sought.

Question 3

Added Mass

This question was difficult and was only answered by seven students. In (a), no students took the hint to consider frames of reference moving at speeds V1 and V2, and no students found another way to show that the boundary conditions were satisfied. Marks were given to students who started in broadly the right direction and explained what they were trying to do. Section (b) was answered perfectly by two students, who used the unsteady Bernoulli equation to integrate p around the surface of the inner cylinder. Some students tried to integrate the kinetic energy over the volume of the fluid, which would probably work but was not successfully shown by anybody. Some students gave excellent physical explanations for the variation of added mass with a2/a1. Part (c) was well answered by six out of seven students, even if they had not answered (a) and (b) well.

Question 4

Flutter

This question was a standard flutter question and was answered well by many students. Around half the students showed a good physical understanding of the process. Sections (a), (b), and (d) were answered well by almost every student. In section (c), most students answered the first part well. Many said (correctly) that the centre of mass should move backwards but did not explain why. In (c) many students saw the relationship with flutter but only around one third explained this well. Common errors were to state that the system would need to be unstable to flutter to obtain thrust, when the reverse is true. No student explained why, or commented that, the centre of mass must be forward. Some students wrote lengthy explanations about thrust and drag on the airfoil at different moments in the cycle but this was not required because it is implicit when describing this motion as the opposite of flutter.

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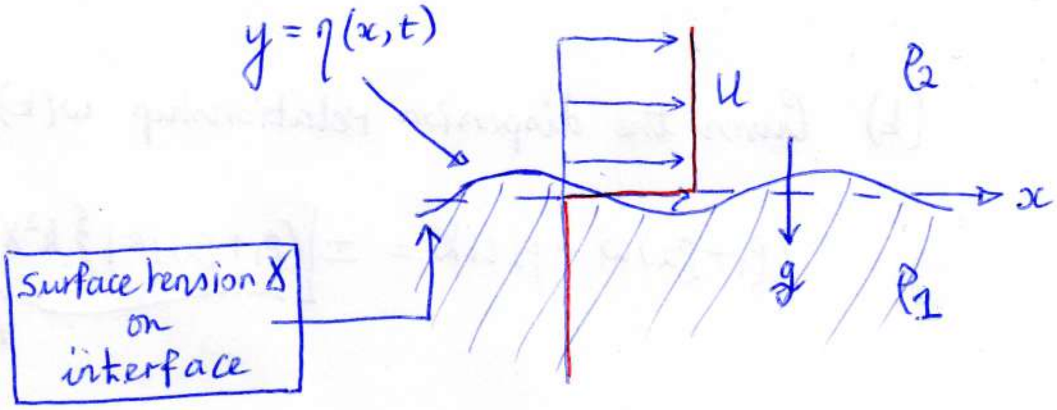


(a) Given frequency $\omega = \omega_R + i\,\omega_I$, $i = \sqrt{-1}$

& a travelling-wave disturbance of form

$$\eta(x,t) = \hat{\eta}\, e^{i(kx-\omega t)}$$

Thus,
$$\begin{aligned} I &= i(kx - \omega t) \\ &= ikx - i\omega t \\ &= ikx - i(\omega_R + i\,\omega_I)t \\ &= i(kx - \omega_R t) + \omega_I t \end{aligned}$$

so that
$$\eta(x,t) = \hat{\eta}\, e^{i(kx-\omega_R t) + \omega_I t}$$

$\therefore$ $\omega_I > 0$ corresponds to exponential growth.

1

(b) Given the dispersion relationship $\omega(k)$ is

$$(\rho_1+\rho_2)\omega - \rho_2 Uk = \pm\Big[\underbrace{(\rho_1+\rho_2)|k|\{k^2\gamma+(\rho_1-\rho_2)g\} - \rho_1\rho_2 U^2k^2}_{A}\Big]^{1/2}$$

Then

$$\omega = \omega_R + i\omega_I = \frac{\rho_2 Uk \pm \left[(\rho_1+\rho_2)|k|\{k^2\gamma+(\rho_1-\rho_2)g\} - \rho_1\rho_2 U^2 k^2\right]^{1/2}}{\rho_1+\rho_2}$$

and for exponential growth the term under the square root must be negative, i.e. we require $A<0$, namely

$$\rho_1\rho_2 U^2k^2 > (\rho_1+\rho_2)|k|\{k^2\gamma+(\rho_1-\rho_2)g\}$$

i.e. $$\frac{\rho_1\rho_2 U^2}{\rho_1+\rho_2} > \frac{|k|}{k^2}\{k^2\gamma+(\rho_1-\rho_2)g\}$$

Given $k^2 = |k||k|$, this is

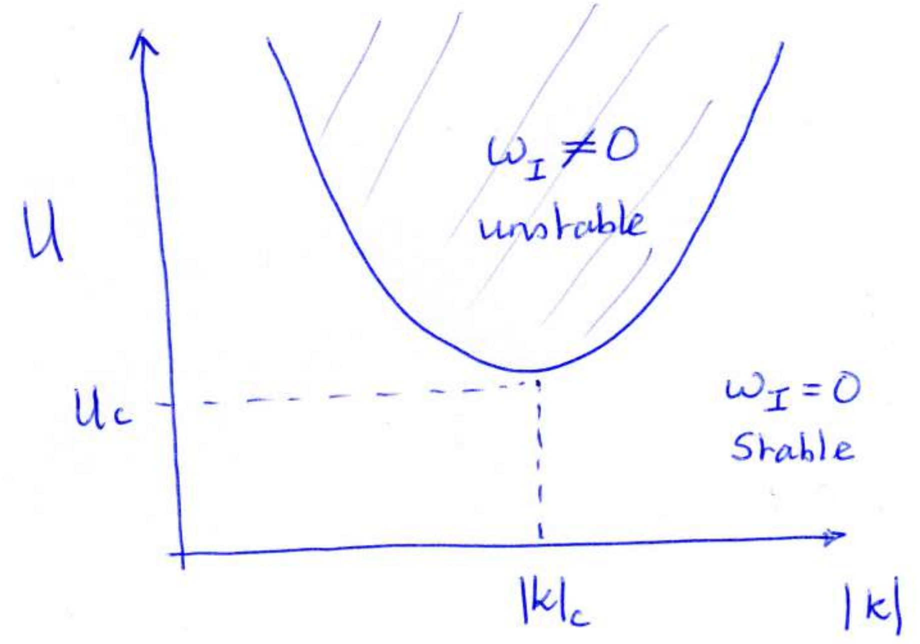
$$\frac{\rho_1\rho_2}{\rho_1+\rho_2}U^2 > (\rho_1-\rho_2)\frac{g}{|k|} + |k|\gamma$$

The magnitude of the RHS varies with k & so we require

$$\frac{\rho_1\rho_2}{\rho_1+\rho_2}U^2 > \min_k\left\{(\rho_1-\rho_2)\frac{g}{|k|} + |k|\gamma\right\} \quad \square$$

(c) Increasing surface tension γ and density difference between the layers $(\rho_1-\rho_2)$ has a stabilising role on small U systems.

1. (d)



(e) To find the first wave to go unstable as U increases, we need find condition that minimises RHS of (1) over $|k|$

Now
$$\frac{d}{d|k|} \text{RHS} = \frac{d}{d|k|}\left\{(\rho_1-\rho_2)\frac{g}{|k|} + |k|\gamma\right\}$$
$$= -(\rho_1-\rho_2)\frac{g}{|k|^2} + \gamma$$

Setting to zero we have the first wave to go unstable, i.e.

$$|k|_c^2 = k_c^2 = \frac{g}{\gamma}(\rho_1-\rho_2)$$

1 (f) We know the first wave to go unstable has

$$k^2 = \frac{g}{\gamma}(\rho_1 - \rho_2)$$

Sub. this into (1) gives

$$\frac{\rho_1 \rho_2}{\rho_1 + \rho_2} U^2 > \frac{(\rho_1 - \rho_2) g}{(\rho_1 - \rho_2)^{1/2} (g/\gamma)^{1/2}} + (\rho_1 - \rho_2)^{1/2} \left(\frac{g}{\gamma}\right)^{1/2} \gamma$$

$$= 2(\rho_1 - \rho_2)^{1/2} (g\gamma)^{1/2}$$

So that

$$U^2 > \frac{2(\rho_1 + \rho_2)(\rho_1 - \rho_2)^{1/2}}{\rho_1 \rho_2} (g\gamma)^{1/2}$$

With
$$\begin{cases} \rho_1 = 1000 \text{ kg/m}^3 \\ \rho_2 = 1.25 \text{ kg/m}^3 \\ \gamma = 0.074 \text{ kg/s}^2 \end{cases}$$

$\underline{U_{crit}^2 = 6.6 \text{ m/s}}$, this, the min. wind speed to drive waves.

2 (a)

Consider a ring of fluid • radius r_1, circumferential velocity u_1

that is displaced outwards to • radius r_2, with circumferential velocity u_2'

Neglecting viscous forces $r_1 u_1 = r_2 u_2'$ as angular mom. conserved

$$\Rightarrow u_2' = \left(\frac{r_1}{r_2}\right) u_1$$

As (given) $-\frac{1}{\rho}\frac{\partial \hat{p}}{\partial r} = -\frac{u_\theta^2}{r}$, the pressure gradient is just sufficient to hold a ring with velocity u_2 at the radius r_2, thus

if $\frac{u_2'^2}{r_2} > \frac{u_2^2}{r_2}$, i.e. $u_2'^2 > u_2^2$ then radial press. grad. is not sufficient to offset the centrifugal force & ring continues outwards (unstable)

Thus require $u_2'^2 \leq u_2^2$ for stability.

Sub. for $u_2' = (r_1/r_2)u_1$ gives

$$r_1^2 u_1^2 \leq r_2^2 u_2^2 \quad \Rightarrow \quad r_2^2 u_2^2 - r_1^2 u_2^2 \geq 0 \qquad (r_1 > r_2)$$

$$\text{i.e.} \quad \frac{d}{dr}(r^2 u^2) \geq 0$$

now $u = r\Omega$

$$\Rightarrow \quad \frac{d}{dr}(r^2\Omega)^2 \geq 0 \quad \text{as req}^{d}.$$

2

b (I) We expect growth rate for jet (capillary) of diameter d, with surface tension γ, $s = s(d, \gamma, \rho, k)$

Now $s \sim [1/T]$

$d \sim [L]$ jet diameter

$k \sim [1/L]$ wave number of perturbation

$\rho \sim [M/L^3]$ jet density

$\gamma \sim [MLT^{-2}/L]$ only quantity with/to include dimensions of time

To obtain dimensionless growth rate, we need multiply by s by quantity with dimensions of time

$$\gamma \sim \left[\frac{M}{T^2}\right] \quad \therefore \quad \gamma^{1/2} \sim \left[\frac{M^{1/2}}{T}\right]$$

Hence, $s\gamma^{-1/2} \sim \left[\frac{1}{T} \cdot \frac{T}{M^{1/2}}\right]$ — (i)

Now $\rho^{1/2} \sim \frac{M^{1/2}}{L^{3/2}}$

Thus, $\frac{s\gamma^{-1/2}}{\rho^{1/2}} \sim \left[\frac{1}{T} \cdot \frac{T}{M^{1/2}} \frac{M^{1/2}}{L^{3/2}}\right]$ so that $\frac{s\rho^{1/2}}{\gamma^{1/2}} d^{3/2} \sim [1]$

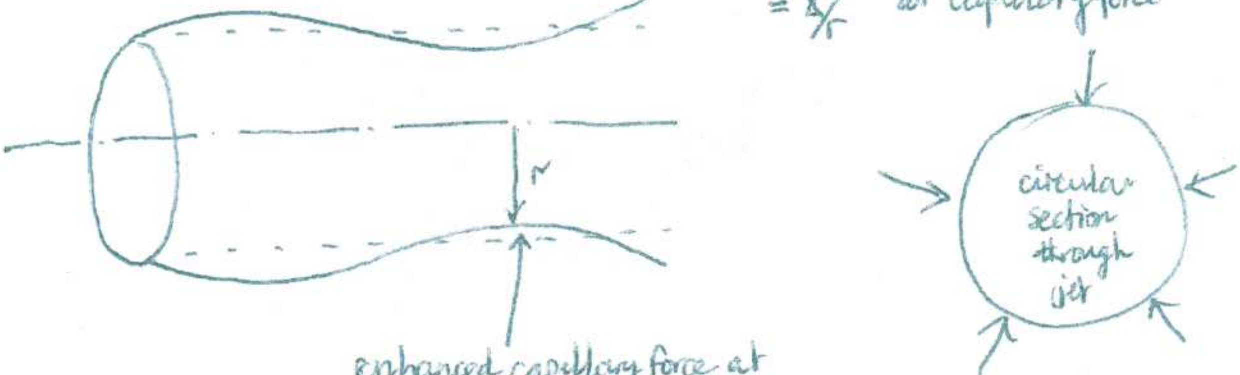
Given dimensionless growth rate depends on dimensionless wavenumber

$$\underline{s\left(\frac{\rho^{1/2} d^{3/2}}{\gamma^{1/2}}\right) = f(ka)}$$

(II) For Jet subject to <u>axisymmetric</u> disturbances:

From the Laplace result $\Delta p = \gamma \nabla \cdot \underset{\sim}{\hat{n}}$ — outward pointing normal

$= \gamma/r$ as capillary force



enhanced capillary force at narrowest region as locally r smaller here $\Rightarrow$ fluid forced away from neck $\Rightarrow$ r decreases $\Rightarrow$ pinchoff

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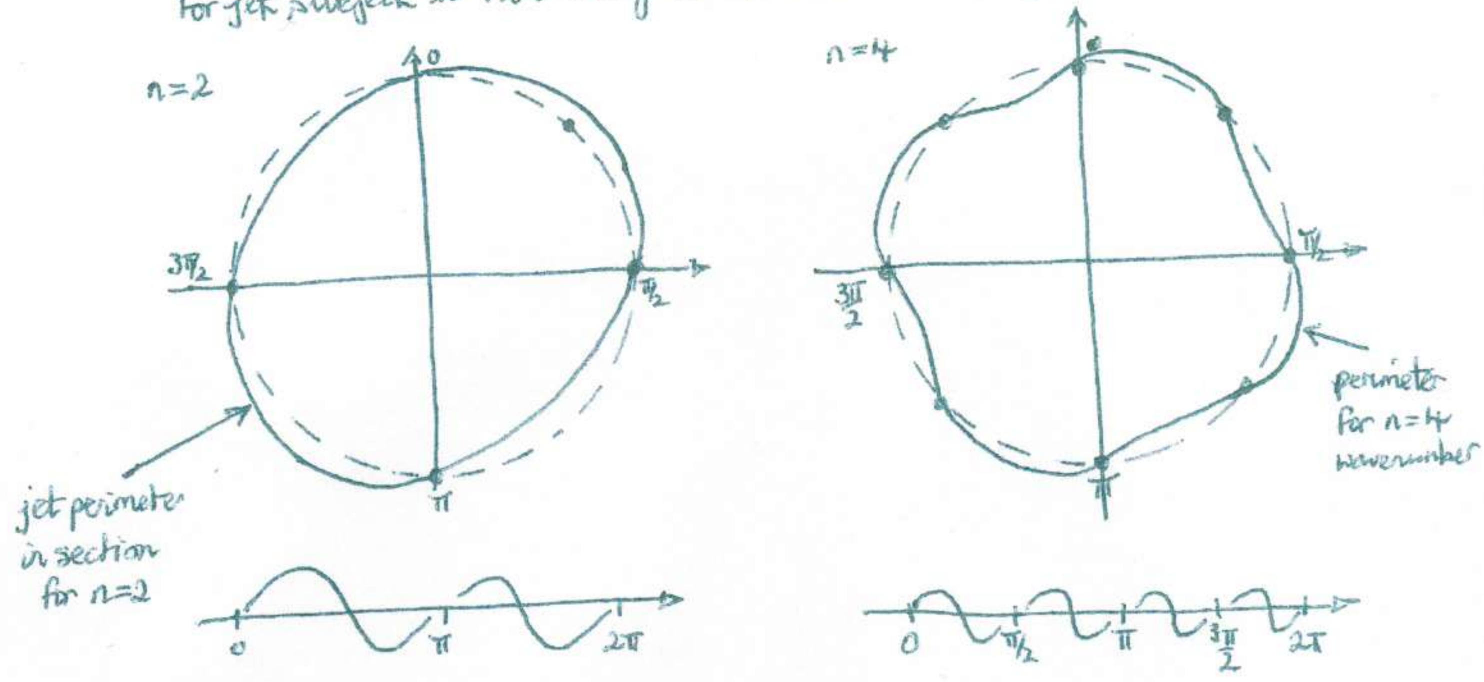
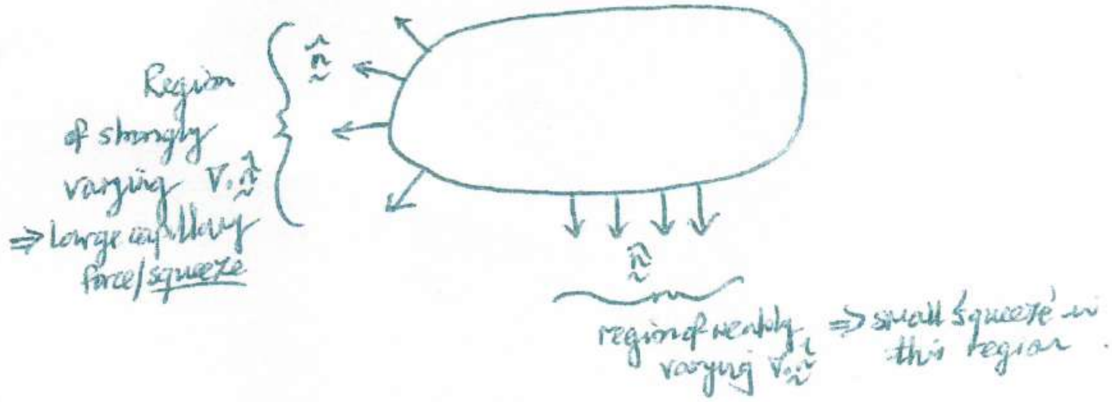
b (ii) In otherwords jet unstable to axisymmetric perturbations

For jet subject to non-axisymmetric disturbances, eg. to mode 2 & mode 4



Regions of perimeter with larger local $\nabla\cdot\hat{n}$ have locally higher capillary forces & tendency to restore section to original circular section, i.e. stabilising. eg.



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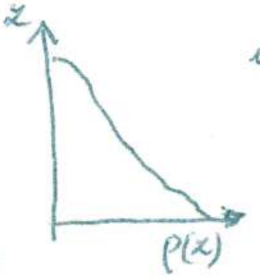
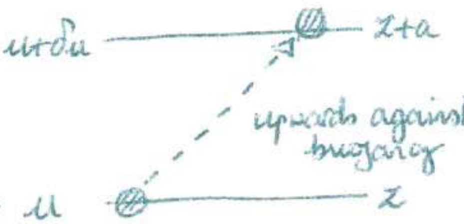
(c) Consider work done in moving particle upward (against buoyancy) — from level z to level $z+a$ — and work done in moving displaced particle down.





Work done against buoyancy

$$I = \int \underset{\sim}{F}_{buoy} \cdot d\underset{\sim}{r}$$

Where buoyancy force on particle of vol. V

$$F_{buoy} = \Big(\underbrace{\rho(z)}_{\text{particle density}} - \underbrace{\Big[\rho(z) + a\frac{d\rho}{dz} + \dots\Big]}_{\text{density of environment}}\Big) gV \approx -(a\, d\rho/dz)\, gV$$ — force felt at general location a

$$\therefore I = \int_{a=0}^{a=\delta z} -a(d\rho/dz)\, gV\, da = \underline{-\tfrac{1}{2}(\delta z)^2\, d\rho/dz\, gV.} \quad (1)$$ — work done in elevating from general position $a=0$ to position $a=\delta z$

Compare this with change in KE associated with particles exchanging place.

$$KE_{before} = \tfrac{1}{2} m_1 u^2 + \tfrac{1}{2} m_2 (u+\delta u)^2, \quad \text{take } m_1 \approx m_2 = \rho_0 V \quad -(2)$$
(i.e. small density variations)

Assume particles take mean vel. on exchanging

$$KE_{after} = \tfrac{1}{2} m_1 \Big[\frac{u+(u+\delta u)}{2}\Big]^2 + \tfrac{1}{2} m_2 \Big[\frac{(u+\delta u)+u}{2}\Big]^2 \quad -(3)$$

Change in KE $= KE_{before} - KE_{after} = \dfrac{\rho_0 V}{4}\, \delta u^2.$ (from (2), (3))

If moving the particles releases energy, this would provide energy to feed instability.

So <u>unstable</u> if $\quad \tfrac{1}{4}\rho_0 V\, \delta u^2 > -\underset{\substack{\uparrow \\ \text{2 particles}}}{2} \times \tfrac{1}{2} \times (\delta z)^2 \dfrac{d\rho}{dz}\, gV \quad$ (using (1))

$$\Rightarrow \quad \tfrac{1}{4} > \frac{-\frac{g}{\rho_0}\frac{d\rho}{dz}}{\left(\frac{du}{dz}\right)^2} \quad \text{for instability}$$

3

(a) In a frame of reference moving at speed V the velocity potential is:

$$\phi = (C_1 - V)\, r\cos\theta + \frac{C_2}{r}\cos\theta$$

The radial velocity is $u_r = \dfrac{\partial\phi}{\partial r} = (C_1 - V)\cos\theta - \dfrac{C_2}{r^2}\cos\theta$

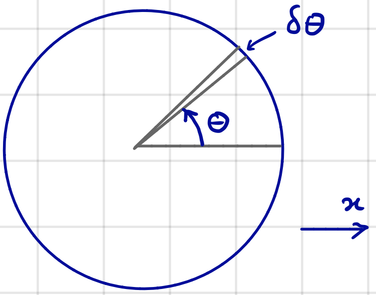
The radius, a, at which $u_r = 0$ is given by

$$a^2 = \frac{C_2}{C_1 - V} = \frac{(V_2 - V_1)\, a_1^2 a_2^2}{(V_2 - V)a_2^2 - (V_1 - V)a_1^2}$$

By inspection, $a = \begin{cases} a_1 & \text{when } V = V_1 \\ a_2 & \text{when } V = V_2 \end{cases}$

These are the required boundary conditions: when moving with the inner cylinder at speed V_1 we require zero velocity across the inner cylinder. When moving with the outer cylinder at speed V_2 we require zero velocity across the outer cylinder.

(b) $V_2 = 0$ and $V_1 \neq 0$. If F is the horizontal force exerted (by the fluid) on the inner cylinder, it can be modelled as an added mass, M, if the force can be expressed as $F = -M\, dV_1/dt$.



$$F = \int_{\theta=0}^{2\pi} -p\cos\theta\, a_1\, d\theta = \int_0^{2\pi} \rho\left(a_1^2\frac{dC_1}{dt} + \frac{dC_2}{dt}\right)\cos^2\theta\, d\theta$$

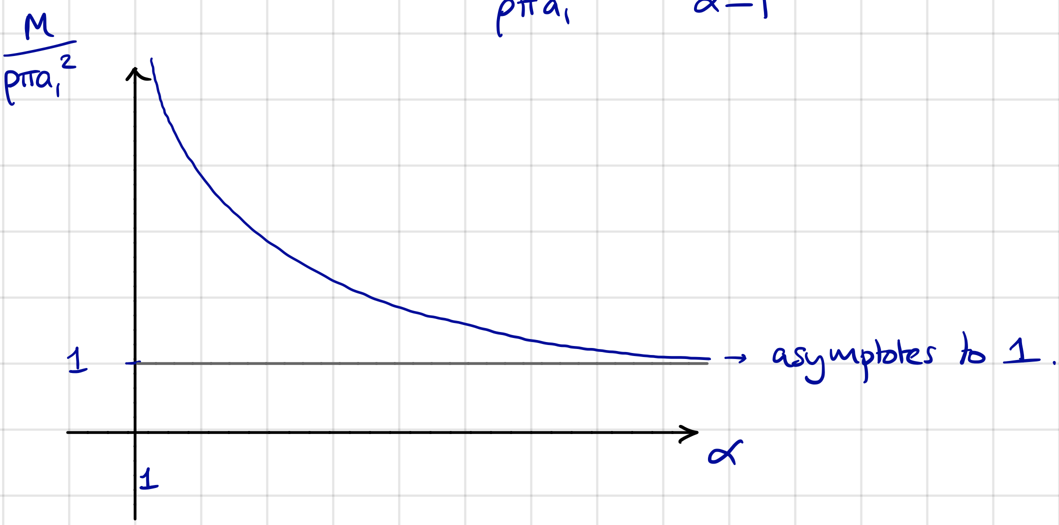
$$= \pi\rho\left(a_1^2\frac{dC_1}{dt} + \frac{dC_2}{dt}\right)$$

If $V_2 = 0$ then $\dfrac{dC_1}{dt} = \dfrac{-a_1^2}{a_2^2 - a_1^2}\dfrac{dV_1}{dt}$ and $\dfrac{dC_2}{dt} = \dfrac{-a_1^2 a_2^2}{a_2^2 - a_1^2}\dfrac{dV_1}{dt}$

$$\Rightarrow F = -\underbrace{\pi\rho\left(\frac{a_1^4}{a_2^2 - a_1^2} + \frac{a_1^2 a_2^2}{a_2^2 - a_1^2}\right)}_{M}\frac{dV_1}{dt}$$

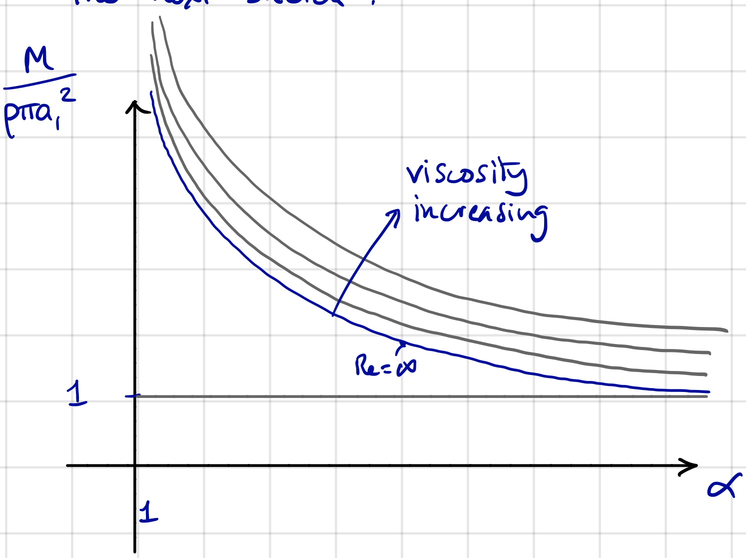
$$\Rightarrow M = \rho\pi a_1^2\left(\frac{a_2^2 + a_1^2}{a_2^2 - a_1^2}\right) = \rho\pi a_1^2\left(\frac{(a_2/a_1)^2 + 1}{(a_2/a_1)^2 - 1}\right)$$

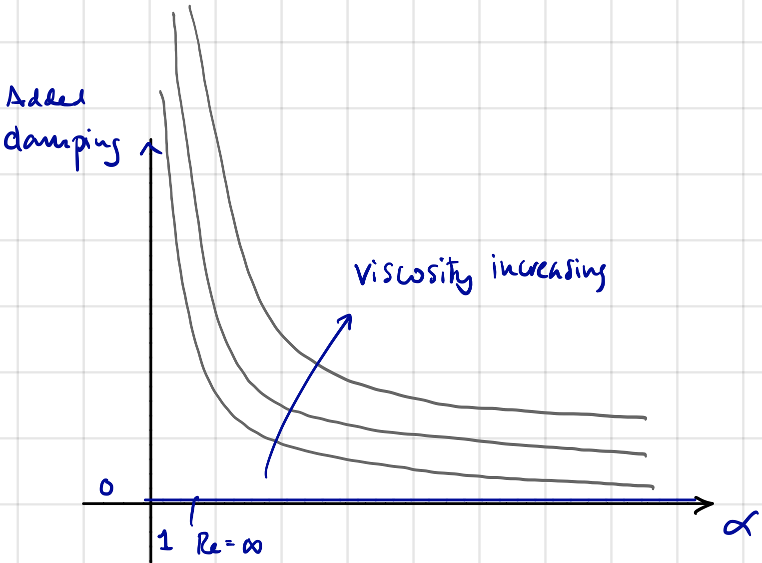
Set $\alpha \equiv (a_2/a_1)$ so that $\frac{M}{\rho \pi a_1^2} = \frac{\alpha^2+1}{\alpha^2-1}$ N.B. $a_2 > a_1$ so $\alpha > 1$.



As $a_2 \to \infty$ the confinement due to the outer cylinder has progressively less influence. The added mass tends towards that of a cylinder in an unconfined fluid : $\rho \pi a_1^2$. As $a_2 \to a_1$ the fluid around the cylinder becomes more confined and has to move faster than it did when unconfined. The required larger change in momentum requires a larger force for a given acceleration dV_1/dt so the added mass in the model increases.

(c) The above sketch is the case for $Re = \infty$ and is a useful baseline for the next sketch.





As the viscosity increases the fluid exerts a larger force on the cylinder for a given dV_1/dt so the added mass increases.

As the viscosity increases the fluid dissipates mechanical energy. This dissipation scales with the velocity gradient so is greater when the velocity is greater, i.e. as $\alpha \to 1$

4(a) Assume $y = Y_0 e^{st}$ and $\theta = \Theta_0 e^{st}$ and substitute into the equations of motion:

$$m s^2 Y_0 - S_x s^2 \Theta_0 + \cancelto{0}{b_y} s Y_0 + k_y Y_0 = q \Theta_0$$

$$I_\theta s^2 \Theta_0 - S_x s^2 Y_0 + \cancelto{0}{b_\theta} s \Theta_0 + k_\theta \Theta_0 = C_a q \Theta$$

where $q = \frac{1}{2} \rho U^2 c \left. \frac{\partial C_L}{\partial \theta} \right|_{\theta = 0}$, which is positive for this shape.

Express as a nonlinear matrix eigenvalue problem:

$$\begin{bmatrix} m s^2 + k_y & -S_x s^2 - q \\ -S_x s^2 & I_\theta s^2 + k_\theta - C_a q \end{bmatrix} \begin{bmatrix} Y_0 \\ \Theta_0 \end{bmatrix} = 0$$

For non-trivial solutions the determinant must be zero:

$$\Rightarrow (m s^2 + k_y)(I_\theta s^2 + k_\theta - C_a q) - S_x s^2 (S_x s^2 + q) = 0$$

$$\Rightarrow s^4 \underbrace{(m I_\theta - S_x^2)}_{\substack{C_0 \\ \text{(positive - see notes)}}} + s^2 \underbrace{(m k_\theta - m C_a q + k_y I_\theta - S_x q)}_{C_2} + \underbrace{(k_y k_\theta - C_a k_y q)}_{C_4} = 0$$

solve for s^2 with the quadratic formula:

$$s^2 = \frac{-C_2 \pm (C_2^2 - 4 C_0 C_4)^{1/2}}{2 C_0}$$

(b) If the system oscillates then s has an imaginary part. If the oscillations grow from rest then s has a positive real part. Therefore s and s^2 must be complex. Therefore $4C_0C_4 > C_2^2$.

(c) When b is positive, the system contains mechanical damping, so some parameter values that were unstable when $b=0$ become stable. The system will be unstable when $4C_0C_4 > C_2^2 + \beta$, where β is some positive number. (β could be derived but this is beyond the scope of this Q).

The system becomes more susceptible to flutter when C_2 decreases. q_r is positive, so this is achieved by increasing S_x, i.e. by moving the centre of mass further backward towards the trailing edge.

(d) The energy comes from the mean flow doing work on the hydrofoil. This increases the drag on the hydrofoil.

(e) When the system is unstable to flutter ($b \to d$), work is dissipated in the dampers and there is increased drag on the hydrofoil. Conversely, when the system is stable to flutter, the dampers could be replaced with actuators in order to do work on the system and produce thrust. The surfer would need to move their centre of mass forward (S_x negative $\Rightarrow C_2$ large $\Rightarrow$ stable to flutter) and then bounce up and down (translational actuator) with one leg slightly earlier than the other (torsional actuator). i.e. classic flutter motion but driven by the actuators rather than by the mean flow. See stills on next page.

4

https://www.youtube.com/watch?v=5XeskRF4jEA