Question ~~2~~ 1 4A10 2014 Gary Hunt

(a) Consider a ring of fluid that is displaced outwards to
- radius r_1, circumferential velocity u_1
- radius r_2, with circumferential velocity u_2'

Neglecting viscous forces $r_1 u_1 = r_2 u_2'$ as angular mom. conserved

$$\Rightarrow u_2' = \left(\frac{r_1}{r_2}\right) u_1$$

As (Given) $-\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{u_\theta^2}{r}$, the pressure gradient is just sufficient to hold a ring with velocity u_2 at the radius r_2, thus

if $\frac{u_2'^2}{r_2} > \frac{u_2^2}{r_2}$, i.e. $u_2'^2 > u_2^2$ then radial press. grad. is not sufficient to offset the centrifugal force & ring continues outwards (unstable)

Thus require $u_2'^2 \leq u_2^2$ for stability.

Sub. for $u_2' = (r_1/r_2)u_1$ gives

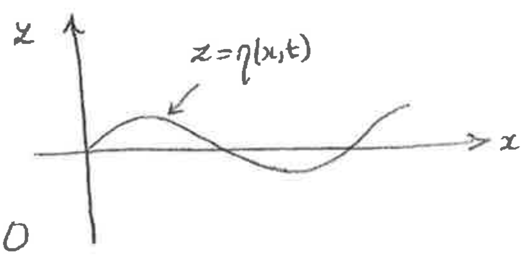
$$r_1^2 u_1^2 \leq r_2^2 u_2^2 \quad \Rightarrow \quad r_2^2 u_2^2 - r_1^2 u_1^2 \geq 0 \qquad (r_1 > r_2)$$

$$\text{ie } \frac{d}{dr}(r^2 u^2) \geq 0$$

now $u = r\Omega$

$$\Rightarrow \underline{\frac{d}{dr}(r^2\Omega)^2 \geq 0} \quad \text{as req}^d.$$

(b). Two-dimensional mixing layer:



For inviscid flow $\nabla \cdot \underset{\sim}{u} = 0$.

Assume irrotational $\underset{\sim}{u} = \nabla\phi$ as $\nabla \times \underset{\sim}{u} = 0$

$\Rightarrow \nabla^2\phi = 0$ Governing eq^n.

Question 1 cont^{d}.

(b) Kinematic b.c:
Particles on interface, remain on interface. So defining $F(x,y,t) = z - \eta(x,t)$ then $F=0$ on $z=\eta$ and $DF/Dt = 0$ gives

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = w \qquad \text{on } z=\eta$$

Dynamic b.c.:
Pressure is continuous across interface/vortex sheet, so from unsteady Bernoulli

$$\frac{p}{\rho} = \left[-\frac{\partial \phi_1}{\partial t} - \frac{\underset{\sim}{u}_1^2}{2} - gz - G_1(t)\right] = \left[-\frac{\partial \phi_2}{\partial t} - \frac{\underset{\sim}{u}_2^2}{2} - gz - G_2(t)\right] \quad \text{on } z=\eta(x,t) \quad *$$

Base state solution is

$$\underline{U} = \begin{cases} U_1 \underset{\sim}{i} & z>0 \\ U_2 \underset{\sim}{i} & z<0 \end{cases} \Rightarrow \begin{array}{ll} \phi_1 = U_1 x & z>0 \\ \phi_2 = U_2 x & z<0 \end{array}$$

$$P = \begin{cases} P_0 - \rho g z & z>0 \\ P_0 - \rho g z & z<0 \end{cases}$$

Introducing perturbations to base state:

$$\phi_1 = \phi_1' + U_1 x, \quad \phi_2 = \phi_2' + U_2 x, \quad p = P + p', \quad \eta = 0 + \eta' \quad \text{\& linearising}$$

Governing equation reduces to $\nabla^2 \phi_1' = 0 \quad z>0, \quad \nabla^2 \phi_2' = 0 \quad z<0$

Boundary conditions reduce to

$$\left.\begin{array}{l} \dfrac{\partial \eta'}{\partial t} + U_1 \dfrac{\partial \eta'}{\partial x} = \dfrac{\partial \phi_1'}{\partial z} \\[2ex] \dfrac{\partial \eta'}{\partial t} + U_2 \dfrac{\partial \eta'}{\partial x} = \dfrac{\partial \phi_2'}{\partial z} \end{array}\right\} \quad \& \quad -U_1 \frac{\partial \phi_1'}{\partial x} - \frac{\partial \phi_1'}{\partial t} = -U_2 \frac{\partial \phi_2'}{\partial x} - \frac{\partial \phi_2'}{\partial t}$$

all applied on $z=0$.

Seek normal mode solutions of form $(\eta', \phi_{(2)}') = (\hat{\eta}, \hat{\phi}_{(2)}')e^{ikx+st}$

* [Moreover, require disturbance to be confined so that $\nabla\phi \to U \underset{\sim}{i}$ as $z \to \pm\infty$]

We obtain

$$\phi_1'(z) = Be^{-kz}e^{ikx+st} \quad \text{on ensuring } \nabla\phi' \to 0 \text{ as } z \to +\infty$$
$$\phi_2'(z) = Ce^{kz}e^{ikx+st} \quad \text{on esuring } \nabla\phi' \to 0 \text{ as } z \to -\infty$$

Kinematic (linearised) b.c.'s give $C = (s + U_2 ik)\hat{\eta}/k$ & $B = -(s + U_1 ik)\hat{\eta}/k$

Dynamic (linearised) b.c. gives $\underline{2s^2 + 2ik(U_1+U_2) - k^2(U_2^2+U_1^2) = 0}$

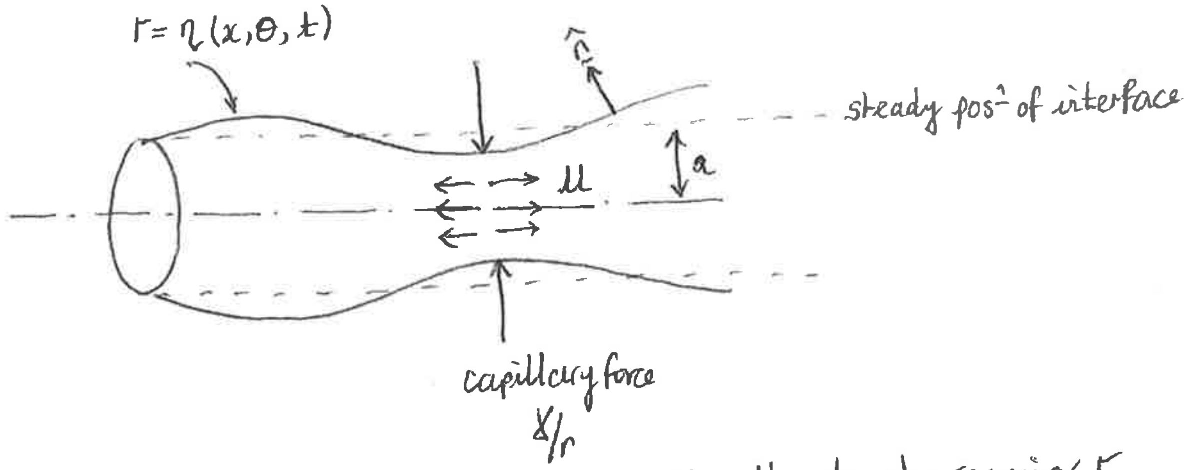
$$\Rightarrow \underline{S = -\frac{ik}{2}(U_1+U_2) \pm 2k(U_1-U_2).}$$

Gary Hunt
2014

Question 2

Physical Mechanism

(a) (i) Laplace's result gives $p - p_\infty = \gamma \nabla . \hat{n}$
$= \dfrac{\gamma}{r}$



The force γ/r forces fluid from the throat, decreasing r and leading to collapse.

(ii) Basic requirement for stability $Re\{s\} < 0$

Given $\quad s^2 = \dfrac{\gamma}{a^3 \rho} \alpha \dfrac{I_n'(\alpha)}{I_n(\alpha)}(1-\alpha^2-n^2) \qquad \alpha = ka$ & $\dfrac{I_n'(\alpha)}{I_n(\alpha)} > 0 \ \forall \alpha$

<u>Non-axisymmetric modes</u> ($n \neq 0$)

$s^2 < 0$ for all α (ie all axial wavenumbers)
i.e. stable to all non-axisymmetric modes.

<u>Axisymmetric modes</u> ($n = 0$)

$s^2 \propto (1-\alpha^2)$

so $s^2 > 0$ for $-1 < \alpha < 1$

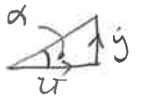
i.e. unstable for $-1 < ka < 1$, given $\lambda = 2\pi/k$,

unstable for wavelengths $-1 < \dfrac{2\pi}{\lambda} a < 1$, ie for $\lambda > 2\pi a$

Conclude: unstable to axisymmetric modes of wavelength exceeding circumference $2\pi a$ of the jet.

4A10 2014 Q3 — Matthew Juniper

a) The apparent angle of attack in a frame of reference moving with the bridge deck is:



$\alpha = \tan^{-1}(\dot{y}/U) \simeq \dot{y}/U$ for small α.

Soft excitation concerns infinitesimal amplitudes, so $\alpha \ll 1$ and therefore $C_y \simeq -\alpha$ (because $\alpha^3 \ll \alpha$ and $\alpha^5 \ll \alpha^3$). The equation of motion becomes $m\ddot{y} + p\dot{y} + ky = qU^2(-\alpha) = -qU\dot{y}$

$$\Rightarrow m\ddot{y} + (p+qU)\dot{y} + ky = 0$$

n.b. U is always positive

The damping term, $p+qU$, is positive for all U, so infinitesimal perturbations are always damped $\Rightarrow$ the bridge is stable to soft excitation

b) The rate of change of energy of the system is: $\dot{E} = m\dot{y}\ddot{y} + ky\dot{y}$

$$\Rightarrow \dot{E} = \dot{y}(qU^2C_y - p\dot{y} - ky) + ky\dot{y}$$ by substitution of the eq. of motion

$$= \dot{y}qU^2C_y - p\dot{y}^2$$

$$= qU^2\left(-\frac{\dot{y}^2}{U} + r\frac{\dot{y}^4}{U^3} - s\frac{\dot{y}^6}{U^5}\right) - p\dot{y}^2$$ by substitution of expression for C_y

$$= -(qU+p)\dot{y}^2 + qr\frac{\dot{y}^4}{U} - qs\frac{\dot{y}^6}{U^3}$$

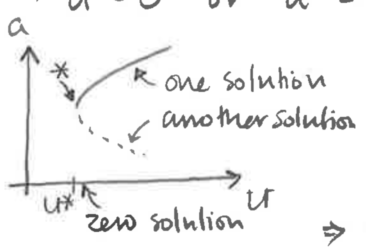
Now, $y = a\cos\omega t$ and $\dot{y} = -\omega a\sin\omega t$. This is substituted into the above expression, which is then integrated over one cycle:

$$\Delta E = \int_0^{2\pi/\omega} \dot{E}\,dt = \underbrace{-\pi p\omega a^2}_{C_1} \underbrace{-\pi q\omega U a^2}_{C_2} + \underbrace{\frac{3\pi qr\omega^3}{4}}_{C_3}\frac{a^4}{U} - \underbrace{\frac{5\pi qs\omega^5}{8}}_{C_4}\frac{a^6}{U^3}$$

c) If the bridge is unstable to hard excitation then there is a periodic solution with non-zero amplitude, a, for which $\Delta E = 0$.

$$\Delta E = a^2\left[-C_1 - C_2U + \frac{C_3}{U}a^2 - \frac{C_4}{U^3}a^4\right] = 0$$

$$\Rightarrow a^2 = 0 \quad \text{or} \quad a^2 = \left[-\frac{C_3}{U} \pm \left(\frac{C_3^2}{U^2} - 4(-C_1 - C_2U)\left(-\frac{C_4}{U^3}\right)\right)^{1/2}\right] \div \left(-2\frac{C_4}{U^3}\right)$$

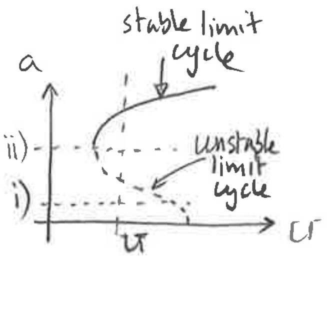


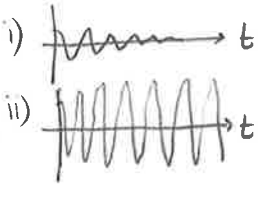
We are interested in the point where these two solutions coincide, marked $*$ on the diagram. This occurs where the term in the square root is zero.

$$\Rightarrow \frac{C_3^2}{U^2} = 4(C_1 + C_2U)\left(\frac{C_4}{U^3}\right)$$

$$\Rightarrow U^* = 4C_1C_4/(C_3^2 - 4C_2C_4)$$

n.b. the critical velocity, U^*, is zero if the mechanical damping, p, is zero.





small impulse: oscillation amplitude dies back to zero amplitude (zero solution).

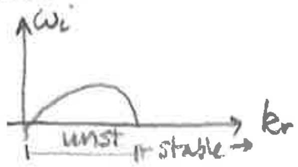
big impulse (bigger than the amplitude of the unstable limit cycle): oscillation amplitude grows to amplitude of stable limit cycle.

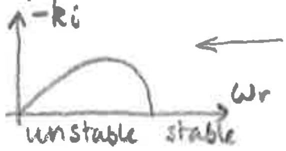
4A10 2014 Q4 Matthew Juniper

(a) The chimneys will have wakes on their downstream sides. These wakes contain regions of absolute instability and therefore are globally unstable and oscillate at defined frequencies. They shed vortices into the flow and the consequent change in momentum of the air causes a fluctuating force on the chimneys at a frequency f_s. If f_s is close to the natural swaying frequency of the chimney, f_n, then f_s will lock onto f_n and force the chimney exactly at its natural frequency. This will cause a large amplitude oscillation. If one chimney lies downwind of the other, it will be hit by vortices exactly at its resonant frequency (the chimneys are identical). This will cause very large oscillations in the down-wind chimney. Strategies to reduce the oscillations include: fairings to disrupt the vortices, changing f_n such that it never is close to f_s, damping. Further details are in the notes

Examiner's comments: This was mostly answered very well. Some students missed out the reduction strategies

(b) Phase velocity = speed of wave crests = ω/k
Group velocity = speed of the envelope of the waves = $d\omega/dk$
where ω is the angular frequency and k = wavenumber
Temporal analysis: examine the temporal growth rate, ω_i, of waves with no spatial growth rate, $k_i = 0$ →





← Spatial analysis: examine the spatial growthrate, $-k_i$, of waves with no temporal growthrate, $\omega_i = 0$

Spatio-temporal analysis: examine all waves - ie for complex k and ω, as if from an impulse. These are most conveniently plotted as contours of $\omega(k)$. Then search for saddle points of $\omega(k)$, at which the group velocity $d\omega/dk = 0$. In an absolutely unstable flow, the temporal growthrate, ω_i, at at least one valid saddle point is positive. In a convectively unstable flow, all valid saddle points have $\omega_i < 0$ but some waves with $d\omega/dk \neq 0$ are unstable. Examples of flows that contain absolute instability are: uniform-density wakes, dense wakes, hot jets, shear flows with surface tension. These all oscillate in the absence of forcing. This question was also answered very well