Solution to Question 1

(a) As instructed, let $\eta = \hat{\eta}\, e^{ikx+st}$ where $(\hat{\eta} \ll 1)$.

Thus: $\frac{\partial \eta}{\partial x} = (ik)\hat{\eta}e^{ikx+st}$ & $\frac{\partial^2 \eta}{\partial x^2} = (ik)^2\hat{\eta}e^{ikx+st} = -k^2\hat{\eta}e^{ikx+st}$.

Hence, the pressure on the surface is:

$$p = \gamma\left(-\frac{\eta}{a^2} - \frac{\partial^2 \eta}{\partial x^2}\right) = \frac{\gamma\hat{\eta}}{a^2}\left((ka)^2 - 1\right)e^{ikx+st}.$$

Note: we are given that the pressure does not vary radially outward across the liquid film and only varies spatially in x. So, if we know the pressure at the liquid-gas interface (i.e. the surface of the film) at a given value of x, then this must be the pressure in the film at this location x. We can substitute this expression for pressure into the governing equation. Governing equation is then

$$\mu\frac{\partial^2 u}{\partial r^2} = \frac{\partial p}{\partial x} = (ik)\frac{\gamma}{a^2}\hat{\eta}\left((ka)^2 - 1\right)e^{ikx+st},$$

$$\implies \quad \frac{\partial^2 u}{\partial r^2} = \underbrace{(ik)\frac{\gamma}{\mu\, a^2}\hat{\eta}\left((ka)^2 - 1\right)e^{ikx+st}} = A.$$

Integrating once gives: $\frac{\partial u}{\partial r} = Ar + f_1$ where $f_1 = const.$ or $f_1 = f_1(x,t)$.

The free surface b.c.: $\mu\frac{\partial u}{\partial r} = 0$ on $r = a + h \implies f_1 = -A(a+h)$.

Integrating again gives: $u = A\frac{r^2}{2} - A(a+h)r + f_2$.

The no-slip b.c.: $u = 0$ on $r = a \implies f_2 = A\left(\frac{a^2}{2} + ha\right)$.

Thus: $u = A\left[\frac{r^2}{2} - (a+h)r + \left(\frac{a^2}{2} + ha\right)\right]$.

By continuity: $\frac{\partial v}{\partial r} = \left(-\frac{\partial u}{\partial x}\right) = -(ik)A\left[\frac{r^2}{2} - (a+h)r + \left(\frac{a^2}{2} + ha\right)\right]$

& integrating gives: $v = -(ik)A\left[\frac{r^3}{6} - (a+h)\frac{r^2}{2} + \left(\frac{a^2}{2} + ha\right)r\right] + f_3$.

Using the no-slip b.c.: $v = 0$ on $r = a$ we have:

$$0 = -(ik)A\left[\frac{a^3}{6} - (a+h)\frac{a^2}{2} + \left(\frac{a^2}{2} + ha\right)a\right] + f_3$$

$$\implies f_3 = (ik)A\left[\frac{a^3}{6} + \frac{a^2 h}{2}\right].$$

Hence:

$$v = -(ik)A\left[\frac{r^3}{6} - (a+h)\frac{r^2}{2} + \left(\frac{a^2}{2} + ha\right)r - \frac{a^3}{6} - \frac{a^2 h}{2}\right].$$

Finally, the kinematic b.c. $\frac{\partial \eta}{\partial t} = v$ on $r = a + h \longrightarrow s\,\hat{\eta}e^{ikx+st} = v|_{r=a+h}$ so that:

$$s\,\hat{\eta}e^{ikx+st} = -(ik)A\underbrace{\left[\frac{(a+h)^3}{6} - (a+h)\frac{(a+h)^2}{2} + \left(\frac{a^2}{2} + ha\right)(a+h) - \frac{a^3}{6} - \frac{a^2 h}{2}\right]}_{=-h^3/3}.$$

Substituting for A, we have:

$$s\,\hat{\eta}e^{ikx+st} = -(ik)(ik)\frac{\gamma}{\mu a^2}\hat{\eta}\left((ka)^2 - 1\right)e^{ikx+st}\left(\frac{-h^3}{3}\right)$$

On rearranging, the growth rate is thus:

$$s = \left(\frac{h}{a}\right)^3 \frac{\gamma}{3\mu a}(ka)^2\left(1 - (ka)^2\right).$$

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[85%]

(b) The solution in part (a) indicates that the liquid film is:

unstable for $(ka)^2 < 1$ [the quantity ka, a dimensionless wavenumber.]

stable for $(ka)^2 > 1$.

In words, the film is unstable to small wavenumber perturbations, i.e. to long waves. Given the wavelength $\lambda = 2\pi/k$, $(ka)^2 < 1 \implies \lambda > 2\pi a$, i.e. the system is unstable to perturbations with wavelengths that exceed the circumference of the liquid film.

The most amplified mode is given by the solution of: $\frac{ds}{d(ka)} = 0$. Substituting for s, this gives $\left(\frac{h}{a}\right)^3 \frac{\gamma}{3\mu a}\left[2(ka) - 4(ka)^3\right] = 0$ and hence $ka = \frac{1}{\sqrt{2}}$ is the wavenumber of the most amplified mode. 3

[15%]

Solution to Question 2

(a) The essence of the approach is as follows: small amplitude perturbations (marked below with a prime) are introduced about the pressure, velocity, etc., of the steady base flow (say, $\underline{u}_0, p_0$) so that

$$\underline{u} = \underline{u}_0 + \underline{u}'(x, y, z, t)$$
$$p = p_0 + p'(x, y, z, t)$$

and substituted into the governing equations of motion and boundary conditions. These perturbations represent the unsteady flow. This system is then linearised, i.e. products of small terms are neglected (as diminishingly small). A given disturbance to the base flow can be Fourier analysed spatially and expressed as an integral sum of normal modes over a range of wavenumbers k. Owing to there being an absence of terms in the governing equations involving products of perturbations, modes are independent/orthogonal and we can solve for the growth rate $s(k)$ by taking a single mode for which k is treated as a parameter - subsequently sweeping through all values of k. Solutions to linearised system therefore sought in terms of normal mode solutions, e.g. $p' = \hat{p}(z)e^{ikx+st}$. If $\text{Real}\{s(k)\} < 0$ for all k, the system is stable. [20%] 4

(b) To investigate the temporal stability of the flow we perform a linear stability analysis.

Two-dimensional mixing layer:

For incompressible flow $\nabla \cdot \underline{u} = 0$

Assume irrotational $\underline{u} = \nabla\phi$ as $\nabla \times \underline{u} = 0 \implies \nabla^2\phi = 0$

Density uniform. Therefore expect this system to be unstable.

Kinematic b.c.: particles on the interface, remain on the interface. So defining $F(x, y, t) = z - \eta(x, t)$ then $F = 0$ on $z = \eta$, and $DF/Dt = 0$ gives:

$$\frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} = w \text{ on } z = \eta.$$

Dynamic b.c.: pressure is continuous across the interface/vortex sheet, so from unsteady Bernoulli:

$$\frac{p}{\rho} = \left[-\frac{\partial\phi}{\partial t} - \frac{u_1^2}{2} - gz - G_1(t)\right] = \left[-\frac{\partial\phi}{\partial t} - \frac{u_2^2}{2} - gz - G_2(t)\right] \text{ on } z = \eta(x, t)$$

Base state solution is:

$$\boldsymbol{u} = \begin{cases} U_1 \boldsymbol{i} & \text{for } z > 0, \\ U_2 \boldsymbol{i} & \text{for } z < 0, \end{cases}$$

$$P = \begin{cases} p_0 - \rho g z & \text{for } z > 0, \\ p_0 - \rho g z & \text{for } z < 0, \end{cases}$$

Introducing perturbations to base state:

$$\phi_1 = \phi_1' + U_1 x, \;\; \phi_2 = \phi_2' + U_2 x, \;\; p = P + p', \;\; \eta = 0 + \eta'$$

Governing equations reduce to: $\nabla^2 \phi_1' = 0$ for $z > 0$ and $\nabla^2 \phi_2' = 0$ for $z < 0$.

Boundary conditions reduce to:

$$\text{on } z = 0 \;\; \begin{cases} \frac{\partial \eta'}{\partial t} + U_1 \frac{\partial \eta'}{\partial x} = \frac{\partial \phi_1'}{\partial z} & \text{on} \quad z = 0^+ \\ \frac{\partial \eta'}{\partial t} + U_2 \frac{\partial \eta'}{\partial x} = \frac{\partial \phi_2'}{\partial z} & \text{on} \quad z = 0^- \end{cases}$$

and

$$-U_1 \frac{\partial \phi_1'}{\partial x} - \frac{\partial \phi_1'}{\partial t} = -U_2 \frac{\partial \phi_2'}{\partial x} - \frac{\partial \phi_2'}{\partial t} \;\text{ on } z = 0.$$

[Moreover, we require the disturbance to be confined, so that $\nabla \phi \to U\mathbf{i}$ as $z \to \pm\infty$.]

Seek normal mode solutions of the form:

$$(\eta', \phi_1', \phi_2') = (\hat{\eta}, \hat{\phi}_1(z), \hat{\phi}_2(z))\, e^{ikx+st}$$

We obtain: $\phi_1'(z) = Be^{-kz}e^{ikx+st}$ ensuring $\nabla\phi_1' \to 0$ as $z \to \infty$
$\phi_2'(z) = Ce^{kz}e^{ikx+st}$ ensuring $\nabla\phi_2' \to 0$ as $z \to -\infty$

Kinematic (linearised) b.c.'s give: $C = (s + U_2\, ik)\frac{\hat{\eta}}{k}$ & $B = -(s + U_1\, ik)\frac{\hat{\eta}}{k}$.

Dynamic (linearised) b.c.'s give: $2s^2 + 2iks(U_1 + U_2) - k^2(U_1^2 + U_2^2) = 0$

Hence:

$$s = -\frac{1}{2}ik(U_1 + U_2) \pm \frac{1}{2}k(U_1 - U_2).$$

16

[80%]

Q3

(a) On Fig. 4 we observe that $\omega_i > 0$ on the k_r axis for both sinuous and varicose waves. This shows that the flow is temporally unstable to both sinuous and varicose waves. The wave with zero group velocity, which is at the saddle points in Fig. 4, is unstable for sinuous waves and stable for varicose waves. Thus we can deduce that the flow is absolutely unstable to sinuous waves and convectively unstable to varicose waves. [6]

If the flow were allowed to evolve from the steady flow shown in Fig. 3, the sinuous motion would dominate with (ω, k) of the saddle point in the left column of Fig 4: $k \simeq 0.5 - 1.2i$ and $\omega \simeq 2.3 + 0.3i$. We therefore expect a wavelength around $2\pi/0.5 = 4\pi$ units.

This was answered well by nearly all candidates. Description of the motion was required, not just the flow's stability

(b) The splitter plate will prohibit sinuous waves. Only varicose waves are permitted and these are not absolutely unstable, although they are convectively unstable. The flow will therefore behave as a weakly-damped oscillator – ie it will respond to external perturbations but will not oscillate by itself. The oscillations will be varicose, not sinuous. [2]

This was answered well by nearly all candidates.

(c) The compression will be zero at the downstream end of the splitter plate and will increase linearly with distance towards the bluff body. Defining $x = 0$ at the downstream end of the bluff body, $T = \tau(x - L)$.

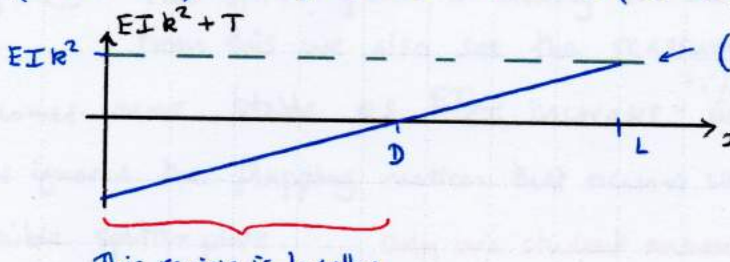
Using methods in the 4A10 course, assume wavy perturbations of the form $Y = Y_0 e^{i(kx - \omega t)}$. Substitute this into the governing equation:

$$EI k^4 + T k^2 - m\omega^2 = 0$$

$$\Rightarrow \quad \omega^2 = \frac{EIk^2 + T}{m} k^2$$

Most students derived this equation but few allowed the tension to be a function of x.

This shows that the splitter plate is <u>locally</u> unstable to buckling when $(EIk^2 + T) < 0$, where $T = \tau(x - L)$ [4]



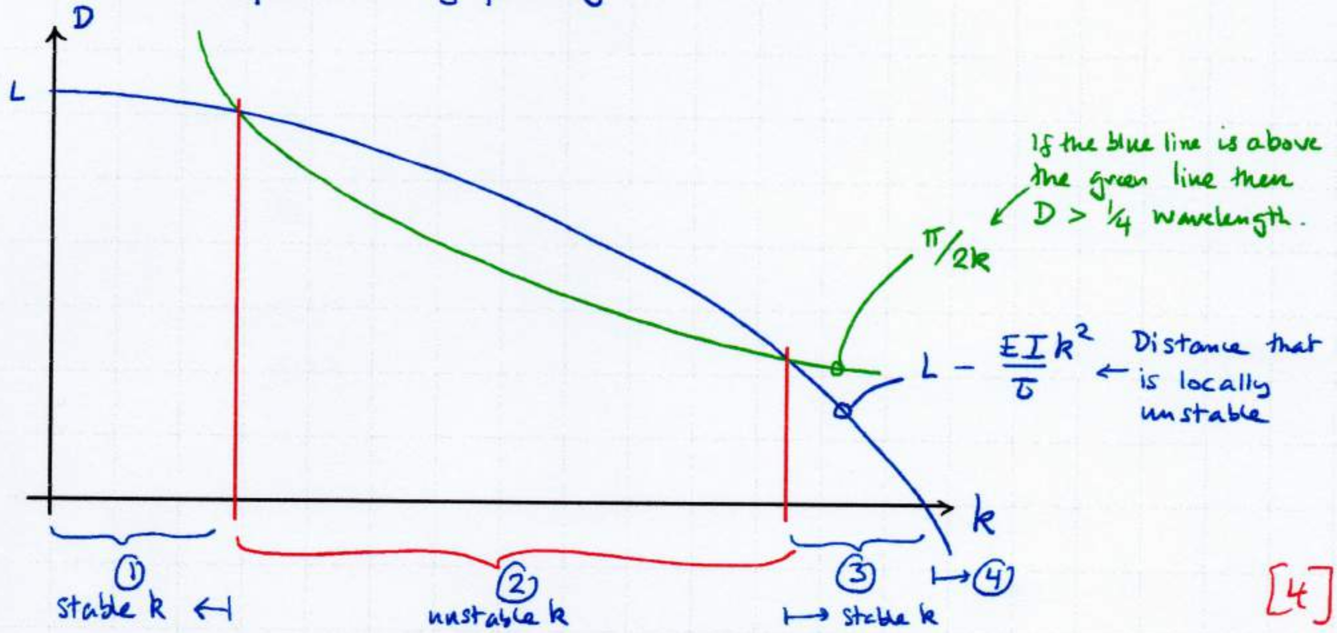
(n.b. we haven't yet said what k is).

This region is locally unstable to buckling.

Only a few students answered this well because this requires tension to be a function of x

We now need to estimate the global behaviour. The distance, D, that is locally unstable is given by: $EIk^2 + \tau(D-L) = 0 \Rightarrow D = L - \frac{EIk^2}{\tau}$.
If we assume that D needs to be around 1/4 wavelength long for the plate to be globally unstable then $D \gtrsim \lambda/4 = 2\pi/4k = \pi/2k$.

This can be represented graphically:



[4]

Region ①: The splitter plate is locally unstable but only to very long wavelengths (small k). These wavelengths greatly exceed the length of the unstable region so the splitter plate is stable to these values of k.

Region ②: For these values of k the distance that is locally unstable exceeds 1/4 wavelength and we can assume (for this question) that the splitter plate is globally unstable over this range of k.

Region ③: the splitter plate is locally unstable to these wavenumbers but only over a very small distance (near the base of the splitter plate). This distance is smaller than 1/4 wavelength so the plate is globally stable to these wavenumbers.

Region ④: the splitter plate is locally stable.

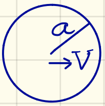
[4]

From this we also see the reassuring behaviour that the splitter plate becomes more stable as EI/τ increases and as L decreases. N.B. this question has ignored the flapping motion that occurs when the sinuous mode interacts with a flexible splitter plate.

Only one student answered this very well. For those who had not considered T to be a function of x, marks were given for intelligent reasoning and physical understanding of the problem

4.

(a)



K.E. of fluid $= \frac{1}{2}\rho V^2 \pi a^2$ per unit length of cylinder

Consider work done in time δt, all increasing the K.E. of the fluid.

$$F V \delta t = \frac{1}{2}\rho\pi a^2\left[(V+\delta V)^2 - V^2\right] \simeq \rho\pi a^2 V\delta V$$

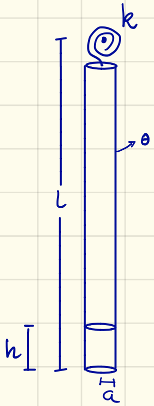
$$\Rightarrow \quad F = \underbrace{\rho\pi a^2}_{\text{added mass due to fluid.}} \overbrace{\frac{\delta V}{\delta t}}^{\text{acceleration of object}}$$

all per unit length.

Most students answered this well.

[4]

(b)



Consider rotational oscillations about the top of the thermometer. The thermometer itself has negligible mass. Therefore the apparent mass of the thermometer comes from the added mass of mercury surrounding the submerged tip. This is $\rho\pi a^2$ per unit length.

$$\Rightarrow \text{total added mass} = \rho\pi a^2 h$$

The eq. of motion is: $(\rho\pi a^2 h)(L - h/2)\ddot{\theta} + k\theta = 0$

But $L \gg h/2 \quad \Rightarrow \quad \rho\pi a^2 h L\ddot{\theta} + k\theta \simeq 0$

Considering $\theta = \theta_0 e^{i\omega_n t} \quad \Rightarrow \quad -\rho\pi a^2 h L\omega_n^2 + k \simeq 0$

$$\Rightarrow \quad \omega_n^2 \simeq \frac{k}{\rho\pi a^2 L h}$$

Most students answered this well.

[6]

(c) Re around the cylinder is $Re = \frac{\rho U D}{\mu} = \frac{13600 \times 0.01 \times 0.01}{1.53 \times 10^{-3}} = 889$

$$\therefore \; St = \frac{f_s D}{U} \simeq 0.2$$

$$\Rightarrow \quad f_s \simeq \frac{0.2U}{D} = \frac{0.2 \times 0.01}{0.01} = 0.2 \text{ Hz}$$

[2]

Almost all students answered this well.

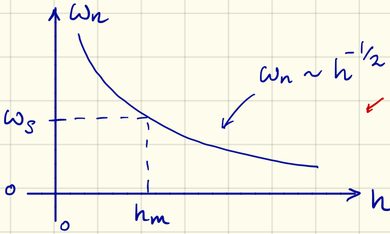
(d) Once vortex shedding starts it will remain at a constant frequency:

$$\omega_s = 2\pi f_s = 2\pi \times 0.2\, U/(2a) = (\pi/5)\, U/a$$

Around half of the students noticed that the vortex shedding would lock on to the cylinder's oscillation. A diagram like this was a good start.

The response is that of a harmonic oscillator with resonant frequency

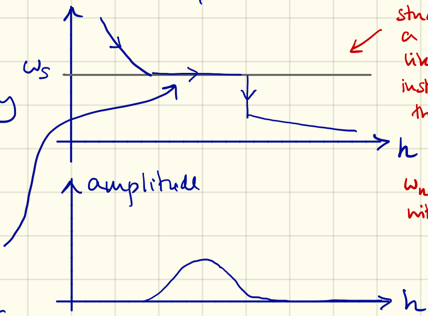
$$\omega_n \simeq (k/\rho\pi a^2 L h)^{1/2}.$$



The cylinder will be forced at its natural frequency when $\omega_n \simeq \omega_s$:

$$\Rightarrow \frac{k}{\rho\pi a^2 L h_m} \simeq \left(\frac{\pi}{5}\right)^2 \frac{U^2}{a^2} \qquad \Rightarrow h_m \simeq \frac{k}{\rho\pi L (\pi/5)^2 U^2}$$

The amplitude of the response will grow and will become sufficiently large that the vortex shedding frequency will lock on to the cylinder's natural frequency, even as h increases further. The oscillation amplitude will be larger around these values of h.

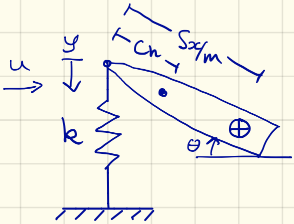


Only a few students drew a diagram like this. Many instead drew the one in the notes, in which ω_n increases with U.

[6]

The model is useful but has many defects. The added mass models the momentum of the mercury around the cylinder as extra momentum (inertia) of the cylinder. The mercury's momentum, however is continually being swept away by the mean flow, meaning that it is inaccurate to model it as momentum of the cylinder, which is not swept away. This loss of momentum would be better modelled as added damping. The model also does not include wave drag or viscous drag. Also the model will become inaccurate if h is no longer much less than L. [2]

A few students answered this very well.



$$m\ddot{y} + S_x\ddot{\theta} + ky = F_y = -q_n\theta \qquad (1)$$

$$I\ddot{\theta} + S_x\ddot{y} = F_\theta = -q_n c_n\theta \qquad (2)$$

(a) $y = Y_0 e^{st}$ and $\theta = \theta_0 e^{st}$; substitute into (1) and (2):

$$(1) \mapsto \quad ms^2 Y_0 + S_x s^2 \theta_0 + kY_0 + q_n\theta_0 = 0$$
$$(2) \mapsto \quad Is^2\theta_0 + S_x s^2 Y_0 + q_n c_n\theta_0 = 0$$

$$\Rightarrow \begin{bmatrix} ms^2 + k & S_x s^2 + q_n \\ S_x s^2 & Is^2 + q_n c_n \end{bmatrix} \begin{bmatrix} Y_0 \\ \theta_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Non-trivial solutions occur when the determinant is zero:

$$(ms^2 + k)(Is^2 + q_n c_n) - (S_x s^2)(S_x s^2 + q_n) = 0$$

$$\Rightarrow s^4\underbrace{\left[mI - S_x^2\right]}_{C_0} + s^2\underbrace{\left[m q_n c_n + kI - S_x q_n\right]}_{C_2} + \underbrace{\left[k q_n c_n\right]}_{C_4} = 0$$

(b) This will be unstable when s contains a real component.

Solving the quadratic for s^2 gives $s^2 = \dfrac{-C_2 \pm \sqrt{C_2^2 - 4C_0C_4}}{2C_0}$

$C_0 = mI - S_x^2 = \dfrac{9S_x^2}{8} - S_x^2 = \dfrac{S_x^2}{8}$, which is positive

C_0 is always positive
C_2 and C_4 can be positive or negative

For the flow to be stable, s must be pure imaginary, so s^2 must be negative and real.

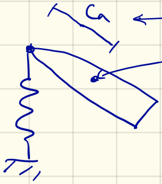
If C_4 is negative, $\sqrt{C_2^2 - 4C_0C_4}$ is positive real and $> C_2$.
Therefore s has one positive real root $\Rightarrow$ instability (torsional divergence)

If C_4 is positive, s^2 will always be negative if $C_2^2 > 4C_0C_4$.

$\therefore$ for stability we require $C_4 > 0$ and $C_2^2 > 4C_0C_4$

[we could ask for a discussion about the instability too]

(c)



c_a ← distance from front of boat to aerodynamic centre.

aerodynamic centre.

Add the aerodynamic force to F_y and F_θ:

$$F_y = -q_h\theta - q_a\theta$$

$$F_\theta = -q_h c_h \theta - q_a c_a \theta$$

$$q_a = \frac{1}{2}\rho_a U_a^2 c \left.\frac{\partial C_{La}}{\partial \theta}\right|_{\theta=0}$$

$$C_4 = k\left[q_h c_h + q_a c_a\right]$$

$$C_2 = m\left[q_h c_h + q_a c_a\right] + kI - S_x\left[q_h + q_a\right]$$

For stability we require $C_4 > 0$ and $C_2^2 > 4C_0C_4$.
q_a will always be +ve for a sail because $\left.\frac{\partial C_{La}}{\partial \theta}\right|_0 > 0$.
Therefore, if q_h is negative, arrange the sail s.t. $q_h c_h < q_a c_a$.

This will avoid torsional divergence.

For $C_2^2 > 4 C_0 C_4$:

$$\left\{ m\left[q_n C_n + q_a C_a\right] + kI - S_x\left[q_n + q_a\right] \right\}^2 - 4 \frac{S_x^2}{8} k \left[q_n C_n + q_a C_a\right] > 0$$

We can only change C_a so let's examine its influence by setting $q_n = 0$:

$$\left\{ m q_a C_a + kI - S_x q_a \right\}^2 - 4 \frac{S_x^2}{8} k q_a C_a > 0$$

always positive.

$$\left\{ kI - m q_a \Big(\underbrace{\frac{S_x}{m} - C_a}_{L} \Big) \right\}^2 - \frac{1}{2} S_x^2 k q_a C_a > 0$$

distance from aerodynamic centre to C of M.

$$\left\{ \frac{9 S_x^2}{8m} k - m q_a L \right\}^2 - \frac{1}{2} S_x^2 k q_a C_a > 0$$

making L large and negative will make the squared term positive. (It will also make the second term a bit bigger, but this is not squared).

This requires C_a large ; i.e. put the sail on the back of the boat.

ENGINEERING TRIPOS PART IIB 2022
MODULE 4A10

Detailed comments

Question 1

Capillary stability of liquid film on a wire

This was a difficult question that, by including a wire, formed an extension of the capillary film instability that candidates studied in class. The governing equations for the perturbations were given and although that these were for the perturbations was italicised, a number of students used up valuable time by substituting for the perturbations into these equations. Candidates that recognised that the pressure only varied along the film did very well, with a number successfully answering all parts perfectly.

Question 2

Temporal linear stability of a mixing layer

This question required the candidates to show that they could perform a full linear stability analysis. Linear stability analysis is a key component of the course and to their credit, the majority of the candidates answered this question well, some answering the question essentially perfectly. It would appear that the candidate scoring the lowest mark ran out of time after attempting only one other question.

Question 3

Linear local stability

This was a difficult question about local stability. Nearly all candidates answered parts (a,b) well, showing a good qualitative understanding of absolute and convective instability. In part (c), nearly all candidates derived a useful dispersion relation but many did not notice that the tension $T(x)$ is a function of position x and therefore missed this aspect of the solution. One student answered (d) very well. For those who had not noticed that T was a function of x, marks were given for intelligent reasoning and physical understanding of the problem.

Question 4

Added mass, vortex shedding, and lock-in

Almost all students answered well parts (a,b), which were about added mass, and part (c), which was about vortex shedding. Many students answered part (d) well, which was about lock-in. They all showed that they had understood the concepts but only around half were able to transfer those concepts to a new situation. Students that answered this question well drew diagrams to illustrate their answer and did not simply repeat the diagrams in the notes. Many students did not attempt to highlight the defects in the model, but a few students answered this part very well.