Version GRH02

EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 26 April 2022 9.30 to 11.10

Module 4A10

FLOW INSTABILITY

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4A10 data sheet (two pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A thin annular layer of liquid, referred to as a liquid film, deposited on a wire of radius a, as depicted in Fig. 1, may become unstable due to surface tension γ and lead to the formation of isolated drops. The thin film, of viscosity μ, has thickness $h + \eta$, where h is the thickness of the undisturbed annular film and $\eta(x, t)$ is an axisymmetric disturbance on the film. In the limit of small film thickness ($h \ll a$) and ignoring gravity, you may assume that the equations governing the evolution of the velocity and pressure *perturbations* of the film are:

$$-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial r^2} = 0 \quad \text{where} \quad p = p(x), \quad \frac{\partial v}{\partial r} = -\frac{\partial u}{\partial x}$$

Here, p denotes pressure, x the horizontal coordinate along the wire, r the radial coordinate, u the horizontal velocity component and v the radial velocity component. The boundary conditions are: no-slip on the wire, i.e. $u = v = 0$ on $r = a$; and the following linearised conditions at the free surface where $r = a + h$:

$$\mu \frac{\partial u}{\partial r} = 0, \quad p = \gamma \left(-\frac{\eta}{a^2} - \frac{\partial^2 \eta}{\partial x^2} \right), \quad \frac{\partial \eta}{\partial t} = v$$

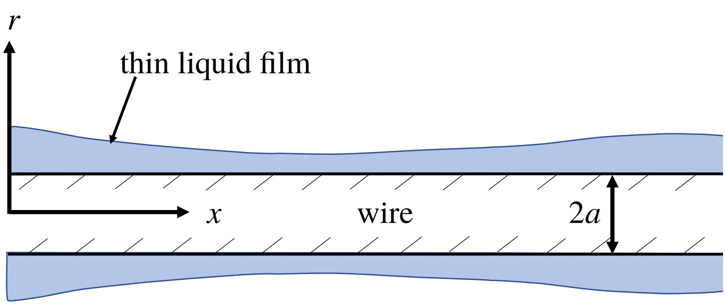

Fig. 1

(a) By introducing normal mode perturbations to the film thickness of the form $\eta = \hat{\eta}\, e^{ikx+st}$, where k is the wavenumber in the x-direction, t is time and $\hat{\eta}$ is a constant, show that the growth rate s of the disturbances is:

$$s = \left(\frac{h}{a}\right)^3 \frac{\gamma}{3\,\mu\, a} (ka)^2 \left(1 - (ka)^2\right)$$ [85%]

(b) Discuss the stability and, in particular, determine the most amplified and the cut-off wavenumbers. [15%]

2 (a) Explain the principles and approach behind a linear stability analysis. Include a description of why a normal mode analysis may be regarded as enabling the influence of all possible small amplitude disturbances to be assessed. [20%]

(b) Consider a two-dimensional inviscid mixing layer with velocity profile

$$\boldsymbol{u} = \begin{cases} U_1 \boldsymbol{i} & \text{for } z > 0, \\ U_2 \boldsymbol{i} & \text{for } z < 0, \end{cases}$$

as shown in Fig. 2.

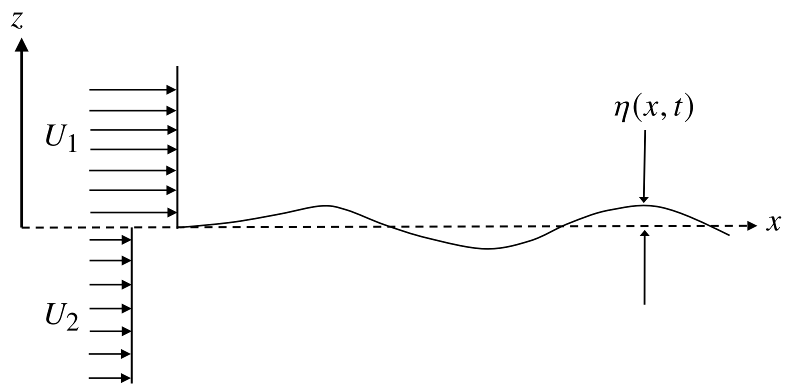

Fig. 2

Investigate the temporal stability of this flow by considering small disturbances to the interface of the vortex sheet of the form

$$\eta = \hat{\eta} e^{ikx+st}$$

to show that the dispersion relationship is given by

$$s = -\frac{1}{2} ik(U_1 + U_2) \pm \frac{1}{2} k(U_1 - U_2).$$ [80%]

3 The inviscid, uniform density, flow behind a bluff body is to be modelled as an unconfined wake as shown in Fig. 3. The two shear layers are 2 units apart and the velocity ratio Λ is equal to -1.4, where $\Lambda \equiv (U_1 - U_2)/(U_1 + U_2)$. Sinuous and varicose waves are considered with wavenumber k and angular frequency ω. Figure 4 shows contours of ω_i (top) and ω_r (bottom) in the complex $k-$plane for sinuous waves (left) and varicose waves (right).

(a) With reference to Fig. 4, determine the stability of this flow and describe its motion if allowed to evolve from the steady flow shown in Fig. 3. [30%]

(b) A splitter plate is added at the line of symmetry such that vertical flow through the line of symmetry is prohibited. With reference to Fig. 4, describe the effect this will have on the flow. [10%]

(c) The splitter plate has Youngs Modulus E, moment of inertia per unit distance into the page I, and mass per unit length m. The reverse flow around the splitter plate exerts a shear stress τ on the splitter plate, causing a (negative) tension T. Defining $x = 0$ at the downstream end of the bluff body, the equation for the vertical displacement, $Y(x,t)$, of the splitter plate is:

$$EI\frac{\partial^4 Y}{\partial x^4} - T\frac{\partial^2 Y}{\partial x^2} + m\frac{\partial^2 Y}{\partial t^2} = 0$$

If the splitter plate has length L, consider the local stability properties of the splitter plate in terms of ω and k, and estimate the length of the plate that is locally unstable to buckling. [20%]

(d) Assume that the plate becomes globally unstable to buckling if the locally unstable length, D, exceeds one quarter wavelength. By considering the local stability as a function of k, and using a graph, sketch and describe qualitatively the global stability of the splitter plate. [40%]

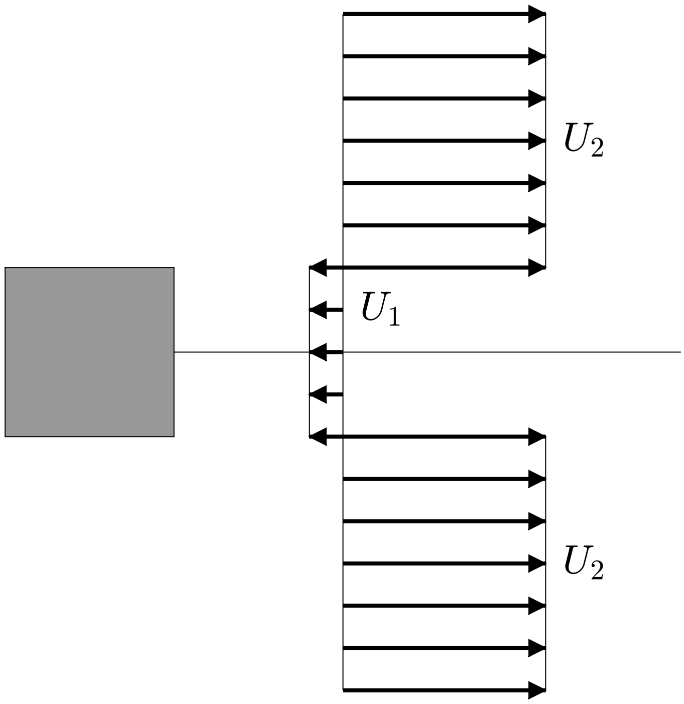

Fig. 3

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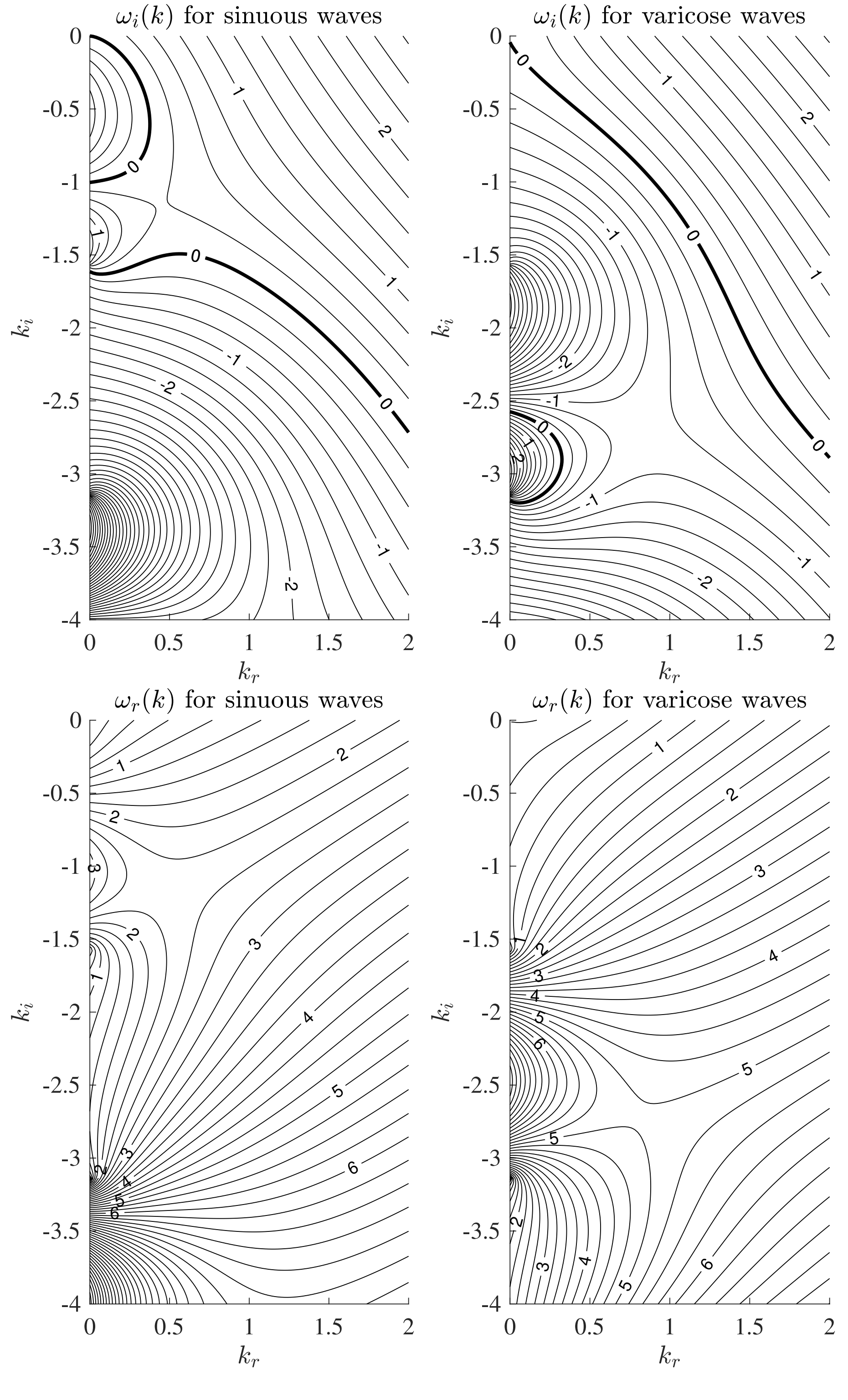
$\omega_i(k)$ for sinuous waves
$\omega_i(k)$ for varicose waves
$\omega_r(k)$ for sinuous waves
$\omega_r(k)$ for varicose waves
k_i
k_r

Fig. 4

4 The *added mass* of an object is the mass that is added to the object in order to account for the influence of the motion of the fluid around the object.

(a) When a cylinder with radius a moves at speed V through an otherwise stationary fluid, the kinetic energy per unit length of the fluid is $\rho V^2 \pi a^2/2$. By considering the work done when accelerating the cylinder, or otherwise, calculate the added mass per unit length of the cylinder. [20%]

(b) In a chemical plant, mercury with density ρ flows down a channel at uniform speed, U. As shown in Fig. 5, a flexible cylindrical thermometer with negligible mass, length l, and radius a is dipped to a depth h into the mercury, where $h \ll l$. The thermometer's flexibility is modelled by a torsional spring at its top, which causes restoring moment $k\theta$. When $U = 0$, show that the natural frequency, ω_n, of vibrations of the thermometer satisfies $\omega_n^2 \approx k/(\rho \pi a^2 l h)$. [30%]

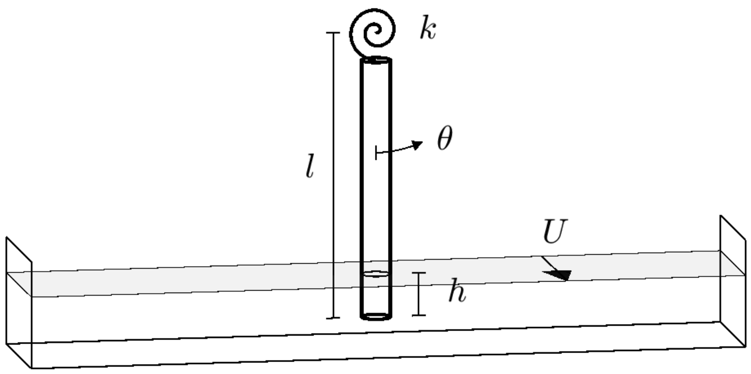

Fig. 5

(c) When flowing, the mercury has $U = 0.01$ ms^{-1}. If $a = 0.005$ m, calculate the frequency of vortex shedding around the cylinder. For mercury, the density is 13600 kgm^{-3} and the viscosity is 1.53×10^{-3} Nsm^{-2}. The Strouhal number of vortex shedding in a flow at speed U around a cylinder of diameter D is approximately $fD/U = 0.2$ for Reynolds numbers from 10^2 to 10^5. [10%]

(d) Model and discuss the behaviour of the thermometer as the depth, h, increases. Discuss any defects in your model. [40%]

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EQUATIONS OF MOTION

For an incompressible isothermal viscous fluid:

Continuity $\nabla \cdot \boldsymbol{u} = 0$

Navier Stokes $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \mu \nabla^2 \boldsymbol{u}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

IRROTATIONAL FLOW $\nabla \times \boldsymbol{u} = \boldsymbol{0}$

velocity potential ϕ,

$$\boldsymbol{u} = \nabla \phi \quad \text{and} \quad \nabla^2 \phi = 0$$

Bernoulli's equation

for inviscid flow $\dfrac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + gz + \dfrac{\partial \phi}{\partial t} =$ constant throughout flow field.

KINEMATIC CONDITION AT A MATERIAL INTERFACE

A surface $z = \eta(x,y,t)$ moves with fluid of velocity $\boldsymbol{u} = (u, v, w)$ if

$$w = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta \quad \text{on } z = \eta(x,t).$$

For η small and $\boldsymbol{u}$ linearly disturbed from $(U,0,0)$

$$w = \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} \quad \text{on } z = 0.$$

SURFACE TENSION σ AT A LIQUID-AIR INTERFACE

Potential energy

The potential energy of a surface of area A is σA.

Pressure difference

The difference in pressure Δp across a liquid-air surface with principal radii of curvature R_1 and R_2 is

$$\Delta p = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

For a surface which is almost a circular cylinder with axis in the x-direction, $r = a + \eta(x,\theta,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = \frac{\sigma}{a} + \sigma\left(-\frac{\eta}{a^2} - \frac{\partial^2 \eta}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 \eta}{\partial \theta^2}\right),$$

where Δp is the difference between the internal and the external surface pressure.

For a surface which is almost plane with $z = \eta(x,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = -\sigma \frac{\partial^2 \eta}{\partial x^2}$$

where Δp is the difference between pressure at $z = \eta^+$ and $z = \eta^-$.

ROTATING FLOW

In steady flows with circular streamlines in which the fluid velocity and pressure are functions of radius r only:

Rayleigh's criterion

The flow is $\begin{matrix}\text{unstable}\\ \text{stable}\end{matrix}$ to inviscid axisymmetric disturbances if Γ^2 $\begin{matrix}\text{decreases}\\ \text{increases}\end{matrix}$ with r.

$\Gamma = 2\pi r V(r)$ is the circulation around a circle of radius r.

Navier Stokes equation simplifies to

$$0 = \mu\left(\frac{d^2 V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2}\right)$$

$$-\rho \frac{V^2}{r} = -\frac{dp}{dr}.$$

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STABILITY OF PARALLEL SHEAR FLOW

Rayleigh's inflexion point theorem

A parallel shear flow with profile $U(z)$ is only unstable to inviscid perturbations if

$$\frac{d^2U}{dz^2}=0 \quad \text{for some } z.$$

CONVECTIVE FLOW

The Boussinesq approximation leads to

$$\nabla\cdot\boldsymbol{u}=0$$

$$\frac{D\boldsymbol{u}}{Dt}=-\frac{1}{\rho_0}\nabla p+(1-\alpha(T-T_0))\boldsymbol{g}+\upsilon\nabla^2\boldsymbol{u}$$

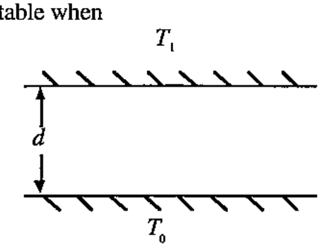
and

$$\frac{DT}{Dt}=\kappa\nabla^2 T$$

Rayleigh-Bénard convection

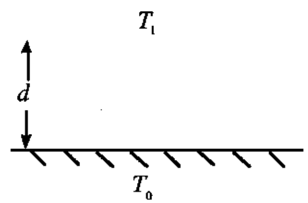
A fluid between two **rigid** plates is unstable when

Ra ≥ 1708



A liquid with a **free** upper surface is unstable when

$$\frac{\mathrm{Ra}}{\mathrm{Ra}_c}+\frac{\mathrm{Ma}}{\mathrm{Ma}_c}\geq 1$$



where

$$\mathrm{Ra}=\frac{g\alpha(T_0-T_1)d^3}{\upsilon\kappa},\quad \mathrm{Ma}=\frac{\chi(T_0-T_1)d}{\rho\upsilon\kappa}\quad \text{with } \chi=-\frac{d\sigma}{dT}$$

$\mathrm{Ra}_c \approx 670$ $\qquad$ $\mathrm{Ma}_c \approx 80.$

USEFUL MATHEMATICAL FORMULA

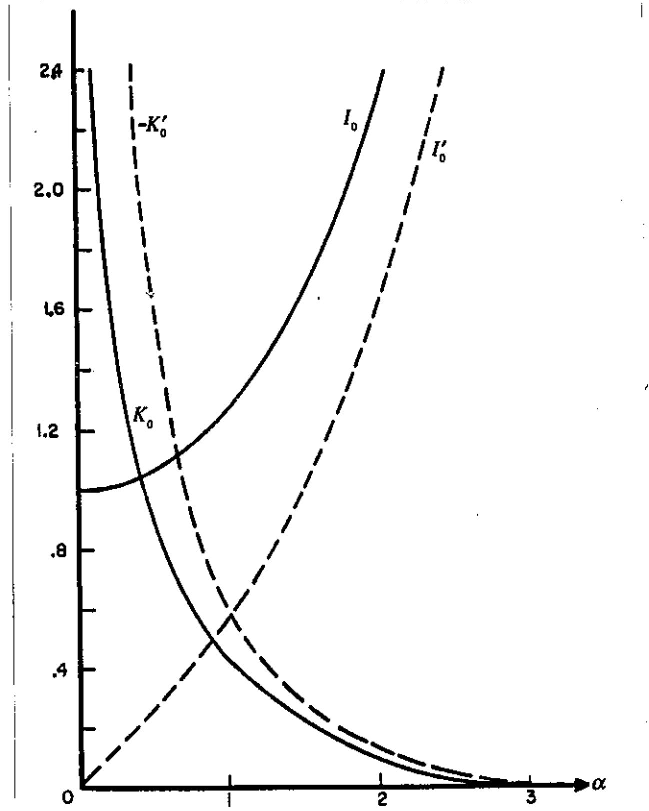
Modified Bessel equation

$I_0(kr)$ and $K_0(kr)$ are two independent solutions of

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-k^2f=0.$$

$I_0(kr)$ is finite at $r=0$ and tends to infinity as $r\to\infty$,

$K_0(kr)$ is infinite at $r=0$ and tends to zero as $r\to\infty$.



$I_0(\alpha)$, $K_0(\alpha)$, $I_0'(\alpha)$, $K_0'(\alpha)$
where $'$ denotes a derivative