4A12 2022 Crib

Q1

(a) $$\begin{cases} (\underset{\sim}{u}\cdot\nabla)\,\underset{\sim}{\omega} = u_z \dfrac{\partial}{\partial z}\underset{\sim}{\omega} = 0 \\ (\underset{\sim}{\omega}\cdot\nabla)\,\underset{\sim}{u} = \omega_x \dfrac{\partial}{\partial x}\underset{\sim}{u} = 0 \end{cases}$$

$$\Rightarrow \quad \frac{\partial \underset{\sim}{\omega}}{\partial t} = \nu\,\nabla^2 \underset{\sim}{\omega} \quad \text{(Diffusion equ.)}$$

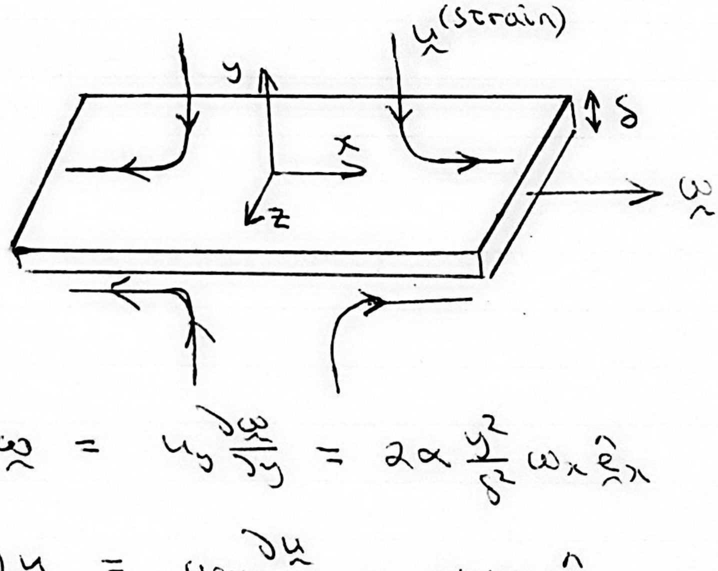
Thickness of sheet diffuses as $\underline{\underline{\delta \sim \sqrt{\nu t}}}$

(diffusion length)

Diffusion cannot create new vorticity, so $\underline{\underline{\Phi = \text{const}}}$.

(b) (i) $\nabla\cdot\underset{\sim}{u}^{(\text{strain})} = \alpha - \alpha = 0$

$$\nabla\times\underset{\sim}{u}^{(\text{strain})} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \\ \alpha x & -\alpha y & 0 \end{vmatrix} = 0$$



(ii) $$\begin{cases} (\underset{\sim}{u}\cdot\nabla)\,\underset{\sim}{\omega} = u_y \dfrac{\partial \underset{\sim}{\omega}}{\partial y} = 2\alpha \dfrac{y^2}{\delta^2}\,\omega_x\,\hat{\underset{\sim}{e}}_x \\ (\underset{\sim}{\omega}\cdot\nabla)\,\underset{\sim}{u} = \omega_x \dfrac{\partial \underset{\sim}{u}}{\partial x} = \alpha\,\omega_x\,\hat{\underset{\sim}{e}}_x \\ \nu\nabla^2\underset{\sim}{\omega} = \nu\dfrac{\partial}{\partial y}\left[-\dfrac{2y}{\delta^2}\,\omega_x\,\hat{\underset{\sim}{e}}_x\right] = -\dfrac{2\nu}{\delta^2}\,\omega_x\hat{\underset{\sim}{e}}_x + 4\nu\dfrac{y^2}{\delta^4}\,\omega_x\hat{\underset{\sim}{e}}_x \end{cases}$$

For steady soln., $(\underset{\sim}{u}\cdot\nabla)\underset{\sim}{\omega} = (\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{u} + \nu\nabla^2\underset{\sim}{\omega}$

$$\Rightarrow \quad 2\alpha\frac{y^2}{\delta_*^2} = \alpha - \frac{2\nu}{\delta_*^2} + 4\nu\frac{y^2}{\delta_*^4}$$

$$\Rightarrow \quad \alpha\left[2\frac{y^2}{\delta_*^2} - 1\right] = \frac{2\nu}{\delta_*^2}\left[2\frac{y^2}{\delta_*^2} - 1\right] \Rightarrow \underline{\underline{\delta_* = \sqrt{2\nu/\alpha}}}$$

(b) (iii) For steady soln., diffusion outward is ballanced by advection inward, and diffusive fall in ω is ballanced by vortex stretching.

(c) (i)
$$\frac{d\delta^2}{dt} + 2\alpha\delta^2 = 4\nu$$

$$\Rightarrow \quad e^{2\alpha t}\frac{d\delta^2}{dt} + 2\alpha e^{2\alpha t}\delta^2 = 4\nu e^{2\alpha t}$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{2\alpha t}\delta^2\right) = 4\nu e^{2\alpha t}$$

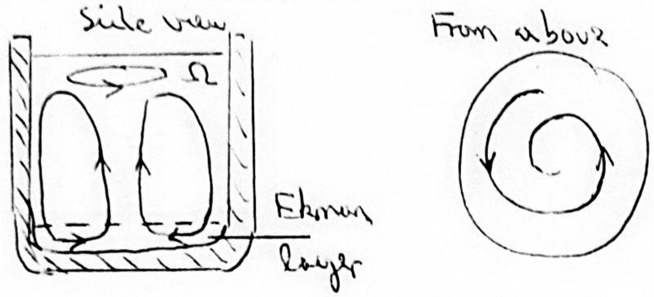
$$\Rightarrow \quad e^{2\alpha t}\delta^2 = \delta_0^2 + \frac{4\nu}{2\alpha}\left[e^{2\alpha t} - 1\right]$$

$$\Rightarrow \quad \underline{\delta^2 = \delta_0^2 e^{-2\alpha t} + \delta_*^2\left[1 - e^{-2\alpha t}\right]}$$

$$\Rightarrow \quad \underline{\underline{\delta^2(t\to\infty) = \delta_*^2}}$$

(ii) If $\delta_0 < \delta_*$, sheet thickens by diffusion, untill $\delta = \delta_*$.
If $\delta_0 > \delta_*$, diffusion is weak, so sheet thins by advection, untill $\delta = \delta_*$.

Q2 (a) (i)



Outside Ekman layer : $\frac{dp}{dr} = \rho \frac{u_\theta^2}{r} \approx \rho \Omega^2 r$.
Same pressure gradient is set up in Ekman layer, where u_θ^2/r is small.
Excess radial pressure gradient drives flow inward within Ekman layer.

(ii) $\delta = f(\nu, \Omega)$ $\Rightarrow$ $\begin{cases} \text{Parameters} = P = 3 \\ \text{Dimensions} = D = 2 \end{cases}$ $\Rightarrow$ Groups $= G = P - 1$
$\uparrow$ m, $\uparrow$ $m^2 s^{-1}$, $\uparrow$ s^{-1}

$= 1$.

Only one dimensionless group. By inspection, $\Pi = \frac{\delta}{\sqrt{\nu/\Omega}}$.

$\Rightarrow$ Π = constant, since no other dimensionless groups for Π to depend on

$\Rightarrow$ $\delta \sim \sqrt{\nu/\Omega}$

(iii) Each fluid particle looses a significant amount of its energy each time it passes through the Ekman layer. So spin-down time is the turnover time of the secondary flow.

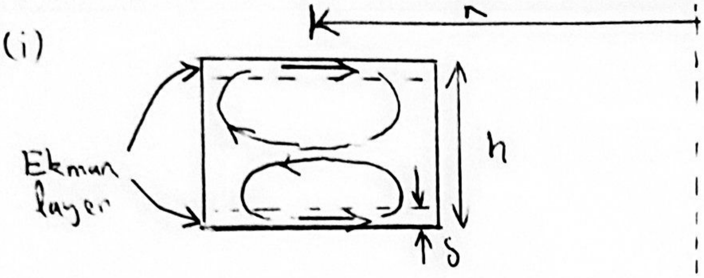
In Ekman layer : $u_r \sim u_\theta \sim \Omega r$

Continuity : $u_z(\pi r^2) \approx (2\pi r) u_r \delta \sim (\pi r^2)\Omega \delta$

$$\Rightarrow u_z \sim \Omega\delta \sim \Omega\sqrt{\frac{\nu}{\Omega}} \sim \sqrt{\nu\Omega}$$

Spindown time $\approx$ turn-over time $\sim \frac{R}{u_z} \sim \frac{R}{\sqrt{\nu \cdot \Omega}}$

(b) (i)



In Ekman layer $u_r \sim \Omega r \sim \Omega R$

Ekman layer thickness is $\delta \sim \sqrt{\frac{\nu}{\Omega}}$

Continuity : $(u_r)_{core}\,(2\pi R)\,h \sim (u_r)_{Ekman}\,(2\pi R)\,2\delta$

$$\Rightarrow \quad (u_r)_{core} \sim \Omega R \frac{2\delta}{h} \sim \underline{\underline{\frac{2R}{h}\sqrt{\nu\Omega}}}$$

(ii) $$\frac{(u_r)_{core}}{u_0} \sim \frac{2\delta}{h} \sim 2\sqrt{\frac{\nu}{\Omega h^2}} \sim 2\sqrt{\frac{\nu}{\Omega R^2}}\,\frac{R}{h} = \frac{20}{\sqrt{10^4}} = \frac{2}{10}$$

$$\text{No. of times} = \frac{\text{through-put time}}{\text{turn-over time of secondary flow}}$$

$$\approx \frac{L / u_0}{h / (u_r)_{core}}$$

$$\sim \frac{L}{h} \times \frac{(u_r)_{core}}{u_0} \sim \frac{300}{5} = \underline{\underline{60}}$$

(iii) Energy loss is the same mechanism as for Tea cup. Each time a fluid particle is flushed through the Ekman layer there is a large loss of energy through viscous dissipation. This energy is replenished by the work done by the pressure forces.

$$\dot{Q}\,(\Delta p)_{duct} = \text{net viscous dissipation} = (\text{dissipation per unit area}) \times L \times 2h$$

$$\Rightarrow \quad \frac{\dot{Q}}{2h}\,\frac{(\Delta p)_{duct}}{L} = \text{dissipation per unit area}$$

$$\Rightarrow \quad \underline{\underline{\left|\frac{dp}{dx}\right| = \frac{2h}{\dot{Q}} \times (\text{dissipation per unit area})}}$$

Q3 (a)

In the inertial subrange, the dissipation rate ε is only a function of the local scale and the kinetic energy that is cascaded from the larger scales. Molecular diffusion is not important. Therefore, the spectrum $E(k)$

E: kinetic energy at wavenumber k per unit mass

k: wavenumber (units: $1/m$)

must be a function of ε & k only.

Units of ε : $m^2 s^{-3}$ (we see this from the usual $\varepsilon = \frac{u^3}{L}$)

Units of k : m^{-1}

From Buckingham's Π-theorem, we can form a single non-dimensional group such that

$$\frac{E}{\varepsilon^m k^n} = C$$

Units of E: We know that $\int_0^\infty E(k)\,dk$ = kinetic energy

$\Rightarrow E$ has units $\left(\frac{m}{s}\right)^2 m = \frac{m^3}{s^2}$

$$\therefore \left.\begin{aligned} 3 &= 2m - n \\ -2 &= -3m \end{aligned}\right\} \Rightarrow \begin{aligned} m &= 2/3 \\ n &= -5/3 \end{aligned}$$

$$\Rightarrow E(k) = C\,\varepsilon^{2/3} k^{-5/3}$$

(b) The usual model for the scalar dissipation is $\varepsilon_\theta = C \frac{\varepsilon}{k} \theta^2$, θ: variance of scalar fluctuations

k: turb kinetic energy

This model is consistent with the concept of the energy cascade; the energy of the large scales, θ^2, is "flowing" to the small scales at a rate determined by the large eddy turnover time K/ε.

Therefore, the rate of dissipation of the scalar fluctuations is

$$\frac{(\text{energy})}{(\text{eddy turnover time})} = C \frac{\varepsilon}{K} \theta^2$$

with the constant C to come from experiment. This model is valid for high Re_t, and when the scalar eddy time $\approx$ velocity eddy timescale.

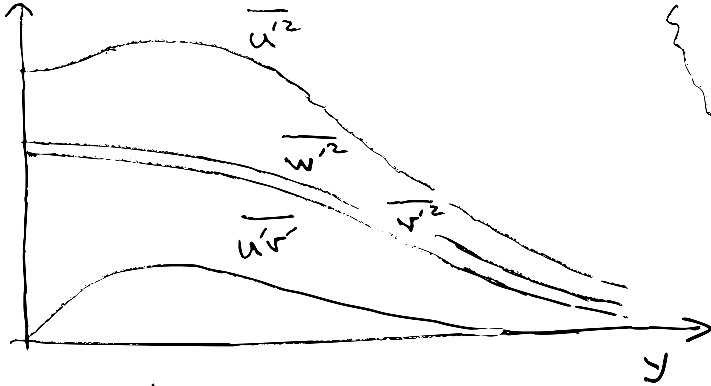
(c) The fastest frequency of the velocity timeseries is expected to be $\left(\frac{\eta_k}{\bar{u}}\right)^{-1}$, where $\bar{u}$ is the mean velocity & η_k the Kolmogorov lengthscale. If the homogeneous isotropic turbulence is characterised by vel fluctuations u' & integral lengthscale L_t, $\eta_k = L_t \, Re_t^{-3/4}$, with $Re_t = \frac{u' L_t}{\nu}$.

Hence, the fastest frequency of the load (which is proportional to the velocity) is

$$\left(\frac{L_t \, Re_t^{-3/4}}{\bar{u}}\right)^{-1}.$$

Q4

(a)



u' = streamwise

v' : cross-stream in y-dir

w' = component in z-dir

Important points to note: $\overline{u'v'}$ is >0

$u' > v', w'$

w' slightly $> v'$

(detail)

$\overline{u'^2}$ max off-axis, $\overline{u'v'} = 0$ on axis (symmetry)

$\overline{u'^2}$ produced by shear, the other components aren't by shear.

As jet expands, the streamwise component of the mean velocity slows down. So the turbulence behaves locally along the axis like a diffuser, i.e. $\overline{u'^2}$ is generated while $\overline{v'^2}$ is destroyed. This can be seen either by an eddy distortion argument (see lecture notes) or by the sign of the production term in the $\overline{u'^2}$ & $\overline{v'^2}$ eqtns :

Q4 (a) cont'd

For $\overline{u'^2}$: Production term $= -\overline{u'^2}\dfrac{d\bar{U}_x}{dx}$. Since $\dfrac{d\bar{U}_x}{dx}<0$,

whole term is –ve.

For $\overline{v'^2}$: Production term $= -\overline{v'^2}\dfrac{dU_y}{dy}$. Since $\dfrac{dU_y}{dy}>0$,

(U_y increases from 0 as y increases), then

whole term is +ve.

Therefore, we expect some anisotropy.

(b) The pressure scrambling terms (or, if we think of the k-eqtn, the pressure diffusion term) imply generation of velocity fluctuations due to pressure fluctuations. These can propagate far and hence can cause vel fluctuations even far away. This mechanism can be more important than generation by shear in the outer parts of a thin shear flow because there the shear is small. Turbulent diffusion of the Reynolds stresses also increase the vel fluctuations in the outer regions. The result is that convection & dissipation balance transport.

Q4 (c)

Since $u_c \sim x^{-1}$ & $\delta \sim x$, using self-similarity gives:

mass flow rate $= \int_0^\infty 2\pi r\, u(r)\, dr = 2\pi \int_0^\infty (\delta\eta)\, u_c F(\eta)\, \delta\, d\eta$

$= 2\pi \delta^2 u_c \int_0^\infty F(\eta)\eta\, d\eta = (\text{Constant}) \cdot x^1$

Momentum flow rate $= \int_0^\infty 2\pi r\, u^2(r)\, dr = 2\pi \int_0^\infty (\delta\eta)\, u_c^2 F^2(\eta)\delta\, d\eta$

$= 2\pi \delta^2 u_c^2 \int_0^\infty F^2(\eta)\eta\, d\eta = (\text{const}) \cdot x^0$

Therefore mass flow rate increases linearly with x & momentum flow rate stays constant. (Actually, the latter is used to find the u_c dependence with x.)