Version EM/2

EGT3
ENGINEERING TRIPOS PART IIB

Thursday 28 April 2022   2 to 3.40

Module 4A12

TURBULENCE AND VORTEX DYNAMICS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book
4A12 Turbulence and Vortex Dynamics Data Card (3 pages).

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 The curl of the shear flow $\mathbf{u}^{(\text{shear})} = u_z(y,t)\hat{\mathbf{e}}_z$ gives the vortex sheet

$$\boldsymbol{\omega}(y,t) = \frac{\Phi}{\sqrt{\pi}\delta}\exp\left[-\left(y/\delta\right)^2\right]\hat{\mathbf{e}}_x ,$$

where $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_z$ are unit vectors associated with the coordinates (*x*, *y*, *z*), $\delta(t)$ is the characteristic thickness of the sheet, and Φ is the flux of vorticity per unit width of the sheet,

$$\Phi = \int_{-\infty}^{\infty} \omega_x dy .$$

(a) Show that the vorticity equation,

$$\frac{D\boldsymbol{\omega}}{Dt} = \left(\boldsymbol{\omega}\cdot\nabla\right)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega} ,$$

reduces to the diffusion equation for this configuration. Explain why, without any mathematics, δ grows as $\delta \sim \sqrt{\nu t}$ while Φ remains a constant. [10%]

(b) This vortex sheet is now immersed in the straining flow $\mathbf{u}^{(\text{strain})} = (\alpha x, -\alpha y, 0)$, where α is a constant.

(i) Show that $\mathbf{u}^{(\text{strain})}$ is both solenoidal and irrotational and sketch the vortex sheet and the straining flow. [10%]

(ii) Confirm that, for an appropriate choice of δ, say $\delta = \delta_*$, the vortex sheet constitutes a steady solution of the vorticity equation and find the relationship between δ_*, α and ν that ensures the vortex sheet remains steady. [40%]

(iii) What physical processes must balance in order to realise this steady configuration. [10%]

(c) The vortex sheet is immersed in the steady straining flow $\mathbf{u}^{(\text{strain})} = (\alpha x, -\alpha y, 0)$ but the initial value of δ is not equal to δ_*. The thickness of the sheet is then a function of time and evolves in accordance with the evolution equation

$$\frac{d\delta^2}{dt} + 2\alpha\delta^2 = 4\nu .$$

(i) Find the general solution of this equation and show that, irrespective of the initial value of δ, say δ_0, the thickness of the sheet tends to δ_* at large times. [20%]

(ii) Explain physically why the thickness of the sheet always tends to δ_*. [10%]

2 (a) A Bödewadt layer forms on the bottom of a teacup containing spinning tea.

(i) Sketch the secondary flow pattern in the Bödewadt layer, both from the side and from above, and explain the physical origin of the secondary flow. [15%]

(ii) Use dimensional analysis to show that the boundary layer thickness is of the order of $\sqrt{\nu/\Omega}$, where Ω is the rotation rate and ν is the kinematic viscosity of the tea. [15%]

(iii) By considering the trajectory of a typical fluid particle, explain why Ekman pumping is the mechanism by which the tea stops spinning. Use continuity to estimate the vertical velocity in the teacup and hence estimate the spin-down time in terms of Ω, ν and the radius of the cup. [15%]

(b) Water is pumped through a helical duct of square cross-section, as shown in Fig. 1. The duct has side h and the inner and outer radii of the duct are $R-h/2$ and $R+h/2$. The pitch of the duct is small, the flow is laminar, and the primary motion can be approximated by $u_\theta = \Omega r$ in cylindrical polar coordinates, where r is the distance from the axis of the helix.

(i) Sketch the secondary flow pattern in the duct and estimate the magnitude of the radial velocity in the core of the duct in terms of Ω, h, R and ν. [15%]

(ii) If $R = 10h$, the length of the duct is $L = 300h$, and the Reynolds number is $\mathrm{Re} = \Omega R^2/\nu = 10^4$, estimate how many times a typical fluid particle is cycled through the Bödewadt layers before leaving the duct. [20%]

(iii) By considering the trajectory of a typical fluid particle within the duct, explain why the viscous dissipation of energy in the duct is dominated by Ekman pumping. Find the relationship between the pressure gradient along the axis of the duct and the net rate of viscous dissipation per unit area of the Bödewadt layers. [20%]

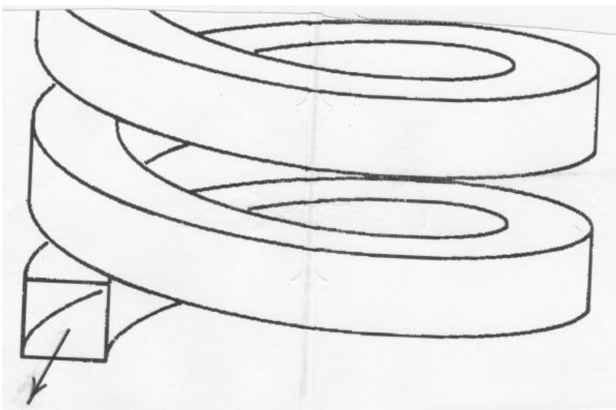

Fig. 1

3 (a) By dimensional arguments or otherwise, derive the decay rate of the turbulent kinetic energy spectrum in the inertial subrange. Discuss carefully the assumptions made. [40%]

(b) Discuss the physical arguments that lead to the model for the scalar dissipation rate given in the Data Card. State clearly the underlying assumptions and discuss their range of validity. [30%]

(c) Assume that the wind approaching a wind turbine pillar is a homogeneous isotropic turbulent flow with unidirectional mean velocity. Ignore the effect of the pillar's structure on the flow and assume that the instantaneous load at a point is proportional to the instantaneous wind speed (mean plus fluctuation). Derive an expression for the expected fastest frequency of the temporal fluctuation of the load. [30%]

4 (a) Consider a planar turbulent jet in stagnant surroundings. Sketch the distributions of the Reynolds stresses across the jet in the self-preserving region. By considering the production terms for the individual normal Reynolds stresses or otherwise, discuss whether we expect the turbulence along the axis of symmetry to be isotropic or not. Explain your reasoning. [40%]

(b) Discuss carefully the reasons for the generation of velocity fluctuations in the outer regions of a thin shear flow. [30%]

(c) Consider a turbulent round axisymmetric jet. The characteristic lengthscale of the jet grows as x^1, where x is the streamwise distance from the nozzle, and the mean centreline velocity decays as x^{-1}. Find how the jet mass and momentum flow rate change with x. [30%]

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Vortex Dynamics Data Card

Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint \boldsymbol{A} \cdot d\boldsymbol{S}$

Stokes : $\int (\nabla \times A) \cdot d\boldsymbol{S} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$

Vector Identities

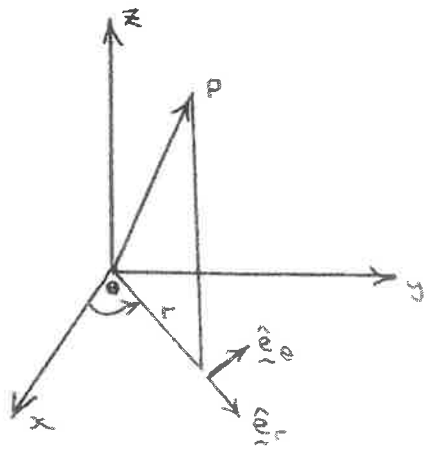
$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B}.\nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})$
$\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot \nabla f$
$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$
$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$

Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$



$$\nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)$$

Cambridge University Engineering Department

4A12: Turbulence

Data Card

Assume incompressible fluid with constant properties.

Continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

Mean momentum:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

Mean scalar:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D \frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

Turbulent kinetic energy ($k = \overline{u_i' u_i'}/2$):

$$\begin{aligned}
\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} \\
&\quad - \overline{u_i' u_j'} \frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'}
\end{aligned}$$

The $k - \varepsilon$ model:

$$\begin{aligned}
\frac{\partial k}{\partial t} + \overline{u}_i \frac{\partial k}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + P_k - \varepsilon \\
\frac{\partial \varepsilon}{\partial t} + \overline{u}_i \frac{\partial \varepsilon}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + c_{\varepsilon 1}\frac{\varepsilon}{k}P_k - c_{\varepsilon 2}\frac{\varepsilon^2}{k}
\end{aligned}$$

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

$$P_k = \frac{1}{2}\nu_t \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)^2$$

$$C_\mu = 0.09,\ c_{\varepsilon 1} = 1.44,\ c_{\varepsilon 2} = 1.92,\ \sigma_k = 1.0,\ \sigma_\varepsilon = 1.3$$

Energy dissipation:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

Scalar fluctuations ($\sigma^2 = \overline{\phi'\phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D \frac{\partial^2 \sigma^2}{\partial x_j^2} - 2\overline{\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'} \frac{\partial \overline{\phi}}{\partial x_j} - 2D \overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

Scalar fluctuations (modelled):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_i \frac{\partial \sigma^2}{\partial x_i} = \frac{\partial}{\partial x_i} \left((D + D_{turb}) \frac{\partial \sigma^2}{\partial x_i} \right) + 2D_{turb} \left(\frac{\partial \overline{\phi}}{\partial x_i} \right)^2 - 2\overline{N}$$

Scalar dissipation:

$$2\overline{N} = 2D \overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2 = 2\frac{u}{L_{turb}}\sigma^2$$

Scaling rule for shear flow, flow dominant in direction x_1:

$$\frac{u}{L_{turb}} \sim \frac{\partial \overline{u}_1}{\partial x_2}$$

Kolmogorov scales:

$$\begin{aligned} \eta_K &= \left(\nu^3/\varepsilon\right)^{1/4} \\ \tau_K &= \left(\nu/\varepsilon\right)^{1/2} \\ v_K &= \left(\nu\varepsilon\right)^{1/4} \end{aligned}$$

Taylor microscale:

$$\varepsilon = 15\nu \frac{u^2}{\lambda^2}$$

Eddy viscosity (general):

$$\begin{aligned} \overline{u_i' u_j'} &= -\nu_{turb} \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) + \frac{2}{3} k \delta_{ij} \\ \overline{u_j' \phi'} &= -D_{turb} \frac{\partial \overline{\phi}}{\partial x_j} \end{aligned}$$

Eddy viscosity (for simple shear):

$$\overline{u_1' u_2'} = -\nu_{turb} \frac{\partial \overline{u}_1}{\partial x_2}$$