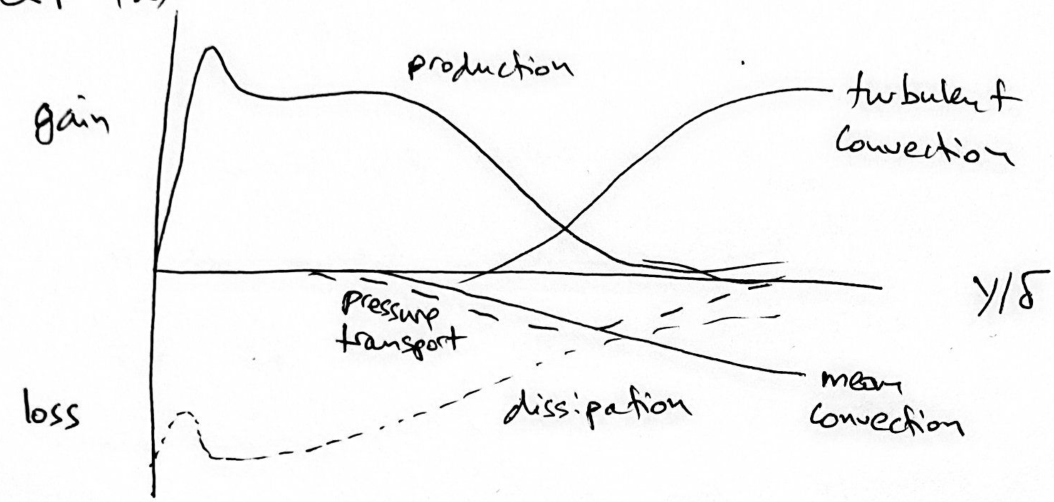
4A12 2023 TURBULENCE & VORTEX DYNAMICS

Q1 (a)



The key features of the budget of k across a boundary layer are:

(i) K ~~bat~~ production balances dissipation for most of the b.l.

(ii) At the outer regions, turbulent diffusion of k from inside the b.l. balances the mean convection.

(iii) pressure diffusion is small but not negligible

(iv) ~~& vi~~ viscous transport is negligible

(v) peak k production is close to the wall.

<u>NOTE</u>: It is not expected to reproduce the exact shapes above, but to identify the key features.

Q1 (b)

Starting from $$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \nu\frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial(\overline{u'v'})}{\partial y} \qquad (1)$$

we employ eddy diffusivity $\nu_{turb} = \kappa u^* y$, which is the correct estimate of diffusivity for a b.l.

For the log-law region, the l.h.s. of the $\bar{u}$ momentum eqtn is zero & the turbulent stresses dominate the viscous ones. Therefore, since $\overline{u'v'} = -\nu_{turb}\frac{\partial \bar{u}}{\partial y}$

$$0 = \frac{\partial}{\partial y}\left(\kappa u^* y \frac{\partial \bar{u}}{\partial y}\right) \qquad (2)$$

$$\Rightarrow \quad \kappa u^* y \frac{\partial \bar{u}}{\partial y} = u^{*2}$$ This is because the term in brackets in Eq(2) is the stress & it is assumed it is equal across the layer & is therefore equal to the value at the wall, i.e. u^{*2}.

Integrating once more $\frac{\bar{u}}{u^*} = \frac{1}{\kappa}\ln\left(\frac{u^* y}{\nu}\right) + A$

after normalizing $\bar{u}$ by u^* & y by ν/u^*.

Since $y^+ = y u^*/\nu$, the log-law of the wall can also be written as $\frac{\bar{u}}{u^*} = \frac{1}{\kappa}\ln y^+ + A$

κ: von Karman constant

Q2 (a) Homogeneous $\rightarrow$ gradients of mean quantities $= 0$

Hence no mean or turbulent transport & no production.

Therefore, $$\frac{d\sigma^2}{dt} = -2\frac{\sqrt{k_0}}{L_0}\sigma^2$$ (k_0 & L_0 are the initial values but we use them ~~for~~ here following the question that k & L stay constant)

$$\Rightarrow \sigma^2 = \sigma_0^2 \exp\left(-\frac{2t}{\left(\frac{L_0}{\sqrt{k_0}}\right)}\right)$$

Therefore $\sigma^2/\sigma_0^2 = 0.1$ when $t/\left(\frac{L_0}{\sqrt{k_0}}\right) = 1.15$

(b) No production of $k \Rightarrow \frac{dk}{dt} = -\varepsilon$.

Using $\varepsilon = \frac{k^{3/2}}{L}$ & keeping $L = L_0$, we have $$\frac{dk}{dt} = -\frac{k^{3/2}}{L_0} \Leftrightarrow \frac{dk}{k^{3/2}} = -\frac{dt}{L_0}$$

$$\Rightarrow k^{-1/2} - k_0^{-1/2} = \frac{t}{2L_0}$$ which can be further manipulated to give the more elegant form:

$$\frac{k}{k_0} = \left[1 + \frac{t}{2T_0}\right]^{-2},$$ where $T_0 = L_0/\sqrt{k_0}$ (the initial turnover time)

NOTE: Using $\varepsilon = \frac{u'^3}{L}$ with $\frac{3}{2}u'^2 = k$ is also acceptable.

Q2 (c) In the $k-\varepsilon$ model from the Data Card, for the case of homogeneous turbulence, the k-eqtn is as in part (b): $\frac{dk}{dt} = -\varepsilon$ (1)

The ε-eqtn becomes $\frac{d\varepsilon}{dt} = -C\frac{\varepsilon^2}{k}$ (2)

where C is some constant (the usually-denoted $C_{\varepsilon 2}$ in the Data Card).

It is difficult to solve (1) & (2) simultaneously analytically. But if we know the decay rate of k (e.g. from part (b)), then (2) can be integrated:

$$\frac{d\varepsilon}{dt} = -C\frac{\varepsilon^2}{k(t)} \Leftrightarrow \frac{d\varepsilon}{\varepsilon^2} = -C\frac{dt}{k(t)}$$

$$\Leftrightarrow \frac{d\varepsilon}{\varepsilon^2} = -\frac{C}{K_0}\left[1+\frac{t}{2T_0}\right]^2 dt \qquad \text{(using part (b))}$$

$$(-1)\left[\varepsilon^{-1} - \varepsilon_0^{-1}\right] = -\frac{C}{K_0}\left[t + \frac{t^2}{2T_0} + \frac{t^3}{12T_0^2}\right]_0^t$$

$$\Leftrightarrow \varepsilon^{-1} = \varepsilon_0^{-1} + \frac{C}{K_0}\left(t + \frac{t^2}{2T_0} + \frac{t^3}{12T_0^2}\right)$$

using $\varepsilon = \frac{k^{3/2}}{L}$ & $\varepsilon_0 = \frac{K_0^{3/2}}{L_0}$, the lengthscale $L = \frac{k^{3/2}}{\varepsilon}$ is evolving by as a function of time

$$\Rightarrow L = \underline{K_0^{3/2}\left(1+\frac{t}{2T_0}\right)^{-3}} \cdot \left[\frac{L_0}{K_0^{3/2}} + \frac{C}{K_0}\left(1 + \frac{t}{2T_0} + \frac{t^2}{12T_0^2}\right)t\right]$$

NOTE: Algebra is getting a little complex - the key point is to recognize the ε-eqtn & the model $\varepsilon = k^{3/2}/L$.

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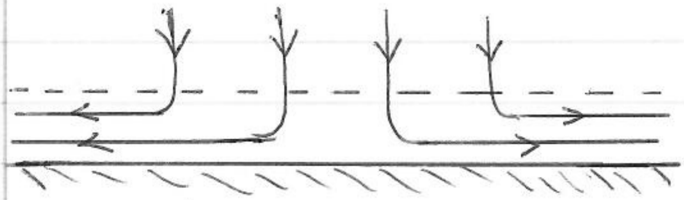
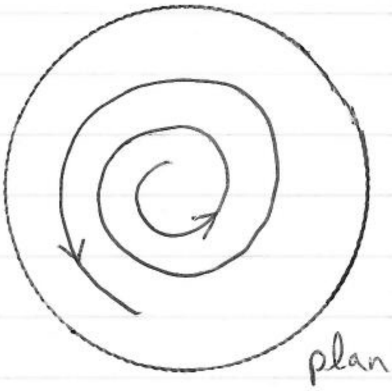
Q1

(a) Karman layer

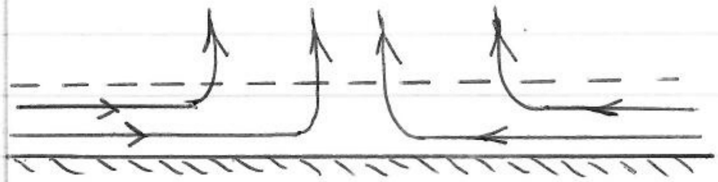


Side view

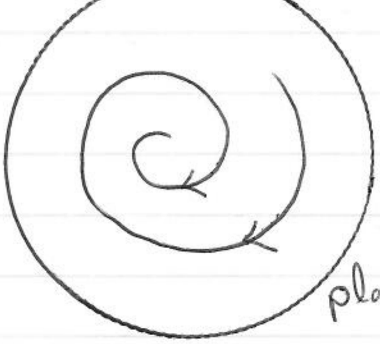


plan view

Bödewadt layer



Side view



plan view

(b) Primary radial force ballance is between the centrifugal force and the radial viscous force:

$$\rho\frac{u_\theta^2}{r} \sim -\rho\nu\nabla^2 u_r \sim \rho\nu\frac{u_r}{\delta^2} \qquad (u_\theta \sim u_r \sim \Omega r)$$

$$\Rightarrow \quad \frac{u_\theta^2}{r} \sim \Omega^2 r \sim \nu\frac{u_r}{\delta^2} \sim \nu\frac{\Omega r}{\delta^2}$$

$$\Rightarrow \quad \underline{\underline{\delta \sim \sqrt{\nu/\Omega}}}$$

(c) Must now include pressure:

$$-\rho\frac{u_\theta^2}{r} \sim -\frac{\partial p}{\partial r} + \rho\nu\nabla^2 u_r \sim -\frac{\partial p}{\partial r} - \rho\nu\frac{u_r}{\delta^2}$$

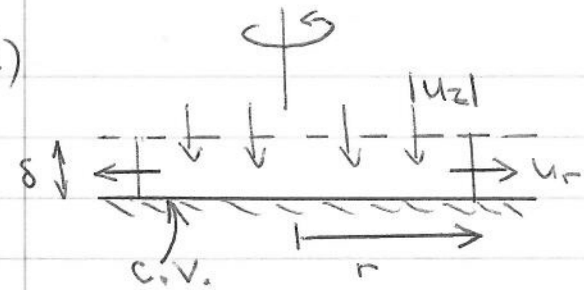
(inside boundary layer)

But outside layer: $\partial p/\partial r = \rho\Omega^2 r$

Combine: $$\underline{\underline{-\nu\frac{u_r}{\delta^2} \sim \frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{u_\theta^2}{r} = \underbrace{\Omega^2 r - \frac{u_\theta^2}{r}}_{\text{positive}}}}$$

Outside layer a +'ve pressure gradient is set up. This pressure gradient is imposed on fluid in the boundary layer, where it exceeds the local centrifugal force. Result is an inflow.

(d)



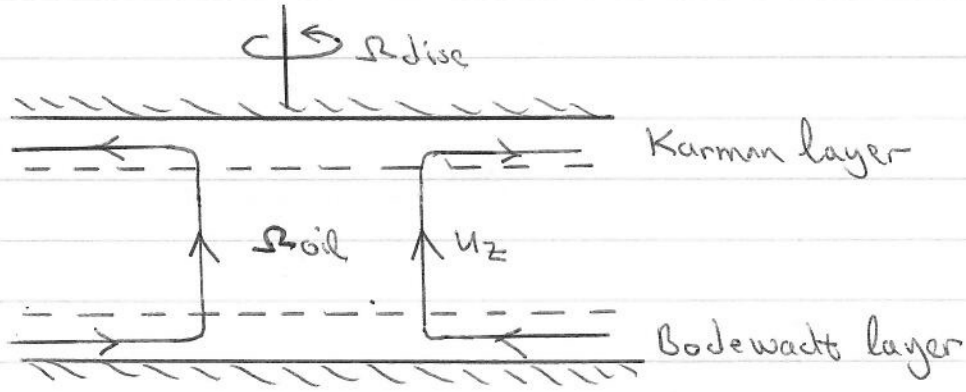
Karman layer

Continuity : $\pi r^2 |u_z| \approx 2\pi r \delta u_r$

$u_r \sim \Omega r \Rightarrow \pi r^2 |u_z| \sim 2\pi r \delta \Omega r$

$\Rightarrow \underline{\underline{|u_z| \sim \Omega \delta \sim \sqrt{\nu \Omega}}}$

(e)



Karman layer : $u_z = 0.885 \sqrt{\nu(\Omega_{disc} - \Omega_{oil})}$

Bodewadt layer : $u_z = 1.35 \sqrt{\nu \Omega_{oil}}$

Continuity : $1.35 \sqrt{\nu \Omega_{oil}} = 0.885 \sqrt{\nu(\Omega_{disc} - \Omega_{oil})}$

$$\Rightarrow \left(\frac{1.35}{0.885}\right)^2 \sim \frac{\Omega_{disc} - \Omega_{oil}}{\Omega_{oil}} = \frac{\Omega_{disc}}{\Omega_{oil}} - 1$$

$$\Rightarrow \frac{\Omega_{oil}}{\Omega_{disc}} = \frac{1}{1 + (1.35/0.885)^2} = \underline{\underline{0.300}}$$

(f) $\tau_{z\theta} = \rho \nu \frac{\partial u_\theta}{\partial z} \sim \rho \nu \frac{\Omega r}{\delta} \quad (\Omega \sim \Omega_{oil} \sim \Omega_{disc})$

$$T = \int_0^R 2\pi r (\tau_{z\theta} r) dr \approx \int_0^R 2\pi r^2 \rho \nu \frac{\Omega r}{\delta} dr \sim \rho \nu \frac{\Omega}{\delta} R^4$$

But $\delta \sim \sqrt{\nu/\Omega} \Rightarrow \underline{\underline{T \sim \rho \sqrt{\nu \Omega}\, \Omega R^4}}$

Q2

(a) $$\frac{D\underset{\sim}{\omega}}{Dt} = \underset{\sim}{\omega}\cdot\nabla\underset{\sim}{u} + \nu\nabla^2\underset{\sim}{\omega}$$

advection of vorticity

vortex stretching term, related to angular momentum conservation

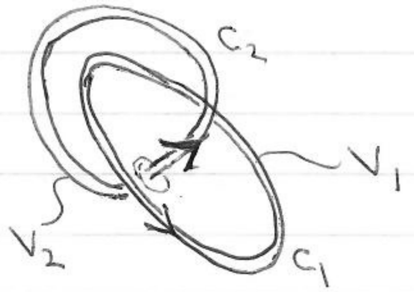
diffusion of vorticity caused by viscous stresses spinning up or spinning down fluid elements

(b) $$\begin{cases} \frac{D}{Dt}d\underset{\sim}{r} = (d\underset{\sim}{r}\cdot\nabla)\underset{\sim}{u} \\ \frac{D\underset{\sim}{\omega}}{Dt} = (\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{u} \end{cases}$$

Same evolution equation, so $\underset{\sim}{\omega}$-lines evolve like dye lines (Helmholtz I)

(c)

(i) $$H = \int_{V_1} \underset{\sim}{u}\cdot\underset{\sim}{\omega}\, dV + \int_{V_2} \underset{\sim}{u}\cdot\underset{\sim}{\omega}\, dV$$



But $\underset{\sim}{\omega}\, dV = \Phi\, d\underset{\sim}{r}$ because

$\Phi = |\underset{\sim}{\omega}| A$ and $dV = A|d\underset{\sim}{r}|$, where $d\underset{\sim}{r}$ is part of C_1 or C_2

Thus $$H = \oint_{C_1} \underset{\sim}{u}\cdot(\Phi_1 d\underset{\sim}{r}) + \oint_{C_2} \underset{\sim}{u}\cdot(\Phi_2 d\underset{\sim}{r})$$

(ii) From Helmholtz II, Φ is constant along a vortex tube, so

$$H = \Phi_1 \oint_{C_1} \underset{\sim}{u}\cdot d\underset{\sim}{r} + \Phi_2 \oint_{C_2} \underset{\sim}{u}\cdot d\underset{\sim}{r}$$

From Stokes

$$\left.\begin{aligned} \oint_{C_1} \underset{\sim}{u}\cdot d\underset{\sim}{r} &= \pm\Phi_2 \\ \oint_{C_2} \underset{\sim}{u}\cdot d\underset{\sim}{r} &= \pm\Phi_1 \end{aligned}\right\}$$

+ if right-handed linkage
− if left-handed linkage

$$\Rightarrow H = (\pm\Phi_1\Phi_2) + (\pm\Phi_2\Phi_1) = \underline{\underline{\pm 2\Phi_1\Phi_2}}$$

If the tubes are not linked, $\oint_{C_1} \underset{\sim}{u}\cdot d\underset{\sim}{r} = \oint_{C_2} \underset{\sim}{u}\cdot d\underset{\sim}{r} = 0$

$$\Rightarrow \underline{\underline{H = 0}}$$

(c) <u>cont.</u>

(iii) Helmholtz II $\Rightarrow$ Φ_1 and Φ_2 are conserved

Helmholtz I $\Rightarrow$ tubes frozen into fluid, so linkage is conserved.

Conservation of Φ_1, Φ_2, linkage $\Rightarrow$ H conserved.

(iv) In a real fluid diffusion allows the linkage of the vortex tubes to change, e.g. the can become unlinked.

(d)
$$\frac{D}{Dt}(\underset{\sim}{u}\cdot\underset{\sim}{\omega}) = \underset{\sim}{u}\cdot\frac{D\underset{\sim}{\omega}}{Dt} + \underset{\sim}{\omega}\cdot\frac{D\underset{\sim}{u}}{Dt}$$

$$= \underset{\sim}{u}\cdot(\underset{\sim}{\omega}\cdot\nabla\underset{\sim}{u}) + \underset{\sim}{\omega}\cdot\left(-\nabla\left(\frac{p}{\rho}\right)\right)$$

$$= \underset{\sim}{\omega}\cdot\nabla\left(\frac{u^2}{2}\right) - \underset{\sim}{\omega}\cdot\nabla\left(\frac{p}{\rho}\right)$$

$$= \underset{\sim}{\omega}\cdot\nabla\left(\frac{u^2}{2}-\frac{p}{\rho}\right)$$

$$= \underline{\underline{\nabla\cdot\left(\left(\frac{u^2}{2}-\frac{p}{\rho}\right)\underset{\sim}{\omega}\right)}} \qquad \left(\text{because } \nabla\cdot\underset{\sim}{\omega}=0\right)$$

Divide all fluid up into material volume elements dV, each of which satisfies $D(dV)/Dt = 0$

$$\frac{D}{Dt}(\underset{\sim}{u}\cdot\underset{\sim}{\omega}\, dV) = \nabla\cdot\left[\underset{\sim}{\omega}\left(\frac{u^2}{2}-\frac{p}{\rho}\right)\right] dV$$

Integrate
$$\frac{d}{dt}\int_{V_\infty} \underset{\sim}{u}\cdot\underset{\sim}{\omega}\, dV = \oint_{S_\infty}\left(\frac{u^2}{2}-\frac{p}{\rho}\right)\underset{\sim}{\omega}\cdot d\underset{\sim}{S}$$

But $\underset{\sim}{\omega}\cdot d\underset{\sim}{S} = 0$ at infinity, so $\frac{d}{dt}\int_{V_\infty}\underset{\sim}{u}\cdot\underset{\sim}{\omega}\, dV = 0$

$$\Rightarrow \underline{\underline{H = \text{const.}}}$$