Version EM/3

EGT3
ENGINEERING TRIPOS PART IIB

Thursday 27 April 2023 2 to 3.40

Module 4A12

TURBULENCE AND VORTEX DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
4A12 Data Card (3 pages)
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) Discuss the budget of the turbulent kinetic energy k across a flat-plate turbulent boundary layer, using appropriate sketches and by reference to the terms in the transport equation for k. [50%]

(b) The simplified governing equation for the streamwise mean velocity $\overline{U}$ in a flat-plate boundary layer is given by

$$\overline{U}\frac{\partial \overline{U}}{\partial x} + \overline{V}\frac{\partial \overline{U}}{\partial y} = \nu \frac{\partial^2 \overline{U}}{\partial y^2} - \frac{\partial \overline{u'v'}}{\partial y}$$

where the symbols have their usual meaning. Starting from the above equation, derive the log-law of the wall for the logarithmic layer. Justify all assumptions and modelling choices you make. [50%]

2 At time $t = 0$, a homogeneous isotropic turbulent flow with zero mean velocity has turbulent kinetic energy $k = k_0$ and integral lengthscale $L = L_0$. The scalar fluctuation variance, σ^2, is initially equal to σ_0^2.

(a) Assuming that k and L remain constant, find the time when the scalar fluctuation variance has decayed to $0.1\sigma_0^2$. [30%]

(b) Assume that the integral lengthscale remains constant. Derive an expression for the decay of k. [30%]

(c) By considering the ε equation from the k–ε model as applied to this problem, and assuming the decay of k in part (b), derive an expression for the evolution of L. [40%]

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3 (a) Sketch the primary and secondary flow patterns associated with the Karman and Bödewadt boundary layers. [10%]

(b) Show that the radial force balance in a Karman layer yields

$$\frac{u_\theta^2}{r} \sim \nu \frac{u_r}{\delta^2}$$

in (r, θ, z) coordinates, where ν is the kinematic viscosity, $\mathbf{u}$ the velocity field, and δ the boundary layer thickness. Use this to estimate the boundary layer thickness on a rotating disc in terms of ν and the disc rotation rate, Ω. [15%]

(c) Write down the equivalent force balance for a Bödewadt layer and hence explain the origin of the secondary flow in a Bödewadt layer. You should assume that the fluid outside the Bödewadt layer rotates uniformly at the rate Ω_{fluid}. [20%]

(d) The axial velocity outside a Karman layer is $|u_z| = 0.885\sqrt{\nu\Omega}$, while that outside a Bödewadt layer is $|u_z| = 1.35\sqrt{\nu\Omega_{fluid}}$. Deduce the scaling law $|u_z| \sim \sqrt{\nu\Omega}$ for a Karman layer, or else $|u_z| \sim \sqrt{\nu\Omega_{fluid}}$ for a Bödewadt layer, using continuity and the results of part (b). [15%]

(e) Two large, parallel discs of radius R share a common axis and the gap between them is filled with oil. The upper disc rotates at Ω_{disc} while the lower disc is stationary. The gap between the discs, h, is much smaller than R but much larger than the thickness of the boundary layers on the discs. Outside the boundary layers, the oil rotates uniformly at the rate Ω_{oil} , which is less than Ω_{disc}, and the flow is laminar. Sketch the secondary flow in the gap and show that $\Omega_{oil} = c\Omega_{disc}$, where c is a constant. What is the numerical value of c? [20%]

(f) Show that the torque, T, transmitted to the lower disc in part (e) scales as $T \sim \rho \sqrt{\nu\Omega_{disc}}\, \Omega_{disc} R^4$. [20%]

4 (a) Write down the vorticity equation for a viscous fluid and briefly explain what each term represents. [10%]

(b) A short line element, $d\mathbf{r}$, which links two material points in the fluid, is governed by the evolution equation

$$\frac{D}{Dt}d\mathbf{r} = (d\mathbf{r} \cdot \nabla)\mathbf{u}$$

where $\mathbf{u}$ is the velocity field. Use this to deduce Helmholtz's first law of inviscid vortex dynamics. [10%]

(c) A vorticity field, $\omega(\mathbf{x}, t)$, in an inviscid fluid consists of two, thin vortex tubes. The tubes are interlinked and they have centrelines C_1 and C_2 and vorticity fluxes Φ_1 and Φ_2. The vortex tubes create a velocity field $\mathbf{u}$ and the net helicity of this flow is defined as $H = \int \mathbf{u} \cdot \omega dV$, where the integral is taken over all space.

(i) Show that H can be rewritten in terms of line integrals, as

$$H = \oint_{C_1} \mathbf{u} \cdot (\Phi_1 d\mathbf{r}) + \oint_{C_2} \mathbf{u} \cdot (\Phi_2 d\mathbf{r})$$

[10%]

(ii) Use Stokes' theorem to show that $H = \pm 2\Phi_1\Phi_2$ and explain when the minus sign is appropriate. Also, show that, if the tubes are not linked, then $H = 0$. [15%]

(iii) Use Helmholtz's two laws of inviscid vortex dynamics to explain why H is an invariant of the motion. [15%]

(iv) In a real fluid, H is only conserved for a short period of time. Explain why this is so. [10%]

(d) Show that, in an inviscid fluid, the helicity density, $\mathbf{u} \cdot \omega$, is governed by

$$\frac{D}{Dt}(\mathbf{u} \cdot \omega) = (\omega \cdot \nabla)\left(\frac{u^2}{2} - \frac{p}{\rho}\right)$$

where p is pressure and ρ is density. Use this to construct an alternative proof that H is conserved. [30%]

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Vortex Dynamics Data Card

Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint \boldsymbol{A} \cdot d\boldsymbol{S}$

Stokes : $\int (\nabla \times A) \cdot d\boldsymbol{S} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$

Vector Identities

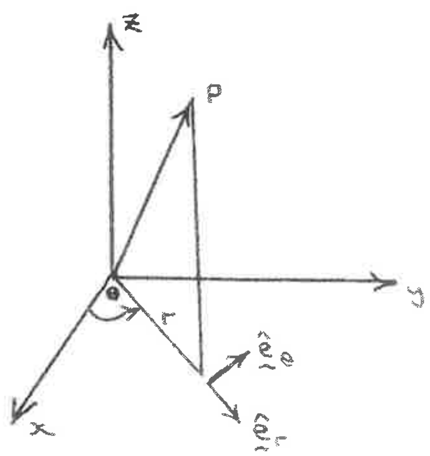
$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B}.\nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})$
$\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot \nabla f$
$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$
$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$

Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \frac{1}{r} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$



$$\nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \right)$$

Cambridge University Engineering Department

4A12: Turbulence

Data Card

Assume incompressible fluid with constant properties.

Continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

Mean momentum:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

Mean scalar:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D \frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

Turbulent kinetic energy ($k = \overline{u_i' u_i'}/2$):

$$\begin{aligned}\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} \\ &\quad - \overline{u_i' u_j'} \frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'}\end{aligned}$$

The $k - \varepsilon$ model:

$$\begin{aligned}\frac{\partial k}{\partial t} + \overline{u}_i \frac{\partial k}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + P_k - \varepsilon \\ \frac{\partial \varepsilon}{\partial t} + \overline{u}_i \frac{\partial \varepsilon}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + c_{\varepsilon 1}\frac{\varepsilon}{k}P_k - c_{\varepsilon 2}\frac{\varepsilon^2}{k} \\ \nu_t &= C_\mu \frac{k^2}{\varepsilon} \\ P_k &= \frac{1}{2}\nu_t \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)^2\end{aligned}$$

$C_\mu = 0.09,\ c_{\varepsilon 1} = 1.44,\ c_{\varepsilon 2} = 1.92,\ \sigma_k = 1.0,\ \sigma_\varepsilon = 1.3$

Energy dissipation:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

Scalar fluctuations ($\sigma^2 = \overline{\phi'\phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D\frac{\partial^2 \sigma^2}{\partial x_j^2} - 2\overline{\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'}\frac{\partial \overline{\phi}}{\partial x_j} - 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

Scalar fluctuations (modelled):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_i \frac{\partial \sigma^2}{\partial x_i} = \frac{\partial}{\partial x_i}\left((D + D_{turb})\frac{\partial \sigma^2}{\partial x_i}\right) + 2D_{turb}\left(\frac{\partial \overline{\phi}}{\partial x_i}\right)^2 - 2\overline{N}$$

Scalar dissipation:

$$2\overline{N} = 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2 = 2\frac{u}{L_{turb}}\sigma^2$$

Scaling rule for shear flow, flow dominant in direction x_1:

$$\frac{u}{L_{turb}} \sim \frac{\partial \overline{u}_1}{\partial x_2}$$

Kolmogorov scales:

$$\begin{aligned} \eta_K &= \left(\nu^3/\varepsilon\right)^{1/4} \\ \tau_K &= \left(\nu/\varepsilon\right)^{1/2} \\ v_K &= \left(\nu\varepsilon\right)^{1/4} \end{aligned}$$

Taylor microscale:

$$\varepsilon = 15\nu\frac{u^2}{\lambda^2}$$

Eddy viscosity (general):

$$\begin{aligned} \overline{u_i'u_j'} &= -\nu_{turb}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij} \\ \overline{u_j'\phi'} &= -D_{turb}\frac{\partial \overline{\phi}}{\partial x_j} \end{aligned}$$

Eddy viscosity (for simple shear):

$$\overline{u_1'u_2'} = -\nu_{turb}\frac{\partial \overline{u}_1}{\partial x_2}$$