EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 24 April 2024 9.30 to 11.10

Module 4A12

TURBULENCE AND VORTEX DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
4A12 Data Card (3 pages)
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) What do we mean by *Taylor's hypothesis* and when is it valid? How does it help us measure the kinetic energy dissipation rate with a single hot wire? [20%]

(b) Consider a homogeneous isotropic turbulent flow with mean streamwise velocity U and no mean velocity in the other directions. Define the integral timescale and the eddy turnover time and discuss the connection between these two quantities. [20%]

(c) Discuss how you would expect the integral lengthscale, the turbulent kinetic energy and its dissipation rate to vary with distance from the grid in wind tunnel turbulence. Give reasons for your answers. [20%]

(d) In wind tunnel turbulence, the integral lengthscale L_t changes with streamwise distance x from the grid as $L_t \sim x^m$. Derive an expression for the evolution of the turbulent kinetic energy k with x. [40%]

2 Consider a planar mixing layer between two fluid streams separated by a splitter plate. One stream has velocity U_0 and the other is stagnant. x_1, x_2 and x_3 are Cartesian coordinates. x_1 is in the direction of the flow, x_2 is normal to the flow and to the plate, and x_3 is normal to the flow and parallel to the plate. The origin is placed at the edge of the splitter plate. We consider downstream distances where the turbulence is fully developed and has reached a self-preserving state.

(a) Sketch qualitatively the mean velocity and the turbulent kinetic energy k versus x_2 at two stations $x_1 = L_1$ and $x_1 = L_2$ with $L_2 > L_1$. What can you say about the ratio $k(x_2 = 0)/U_0$ as a function of x_1? Justify your answer. [20%]

(b) Sketch carefully the distribution of the normal stresses $\overline{u_1'^2}$, $\overline{u_2'^2}$, and $\overline{u_3'^2}$, and of the Reynolds stress $\overline{u_1' u_2'}$ as a function of x_2. Give reasons for the relative magnitudes of the normal stresses and the sign of the Reynolds stress. [40%]

(c) Write down the mean continuity, cross-stream momentum, and streamwise momentum equations for the above flow. Compare the order of magnitude of the various terms and hence simplify the streamwise momentum equation. Show that the half-width δ must scale as x_1. [40%]

3 At the origin, the velocity of the line vortex

$$u_\theta = \frac{\Gamma_0}{2\pi r}$$

becomes infinite. Such behaviour is prohibited because continuity requires that the velocity be zero at the origin. To meet this requirement we must have a core region where the flow is rotational and viscous forces are important. We assume that the velocity $u_\theta(r,t)$ is a function of r and t only and $u_r = u_z = 0$. The θ-direction momentum equation simplifies to:

$$\frac{\partial u_\theta}{\partial t} = \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right) ,$$

where ν is the kinematic viscosity of the fluid. Boundary conditions are

$$\begin{aligned} &u_\theta(r=0,t) = 0 , \\ &u_\theta(r\to\infty,t) = \tfrac{\Gamma_0}{2\pi r} , \\ &u_\theta(r>0,t=0) = \tfrac{\Gamma_0}{2\pi r} . \end{aligned}$$

(a) Explain why a similarity solution may exist. [10%]

(b) It is convenient to take the circulation

$$\Gamma(r,t) = 2\pi r u_\theta(r,t)$$

as the dependent variable of the problem. Deduce the equation for Γ. [20%]

(c) We look for a similarity solution of the form

$$\Gamma = f(\eta) , \quad \eta = r/(\nu t)^{1/2} .$$

Derive an ordinary differential equation for f. [30%]

(d) Determine f with appropriate boundary conditions and hence u_θ. [20%]

(e) Comment on the behaviour of u_θ when $r^2/(\nu t)$ is very large and very small. [20%]

4 (a) Prove Helmholtz's first theorem: the fluid elements that lie on a vortex line at some initial instant continue to lie on that vortex line for all time, i.e. the vortex lines move with the fluid. [20%]

(b) Figure 1 shows a river entering a bend. The bulk of the upstream flow is irrotational and viscous effects in the bend can be neglected.

(i) How does the flow speed depend on the radial position r? Justify your conclusion.

(ii) Assume that the bend is circular. After a short transition region the flow becomes independent of the azimuthal angle, θ. Show that the streamlines are circular.

(iii) How precisely does the flow speed depend on the radial position r in (ii)?

[40%]

(c) On the bed of the river there is a rotational, boundary layer region.

(i) Indicate on a sketch how the vortex lines associated with this region evolve as the river traverses the bend. Describe the resulting secondary flow. What is the weakness in this argument?

(ii) Use a radial force balance argument to deduce the secondary flow pattern. [40%]

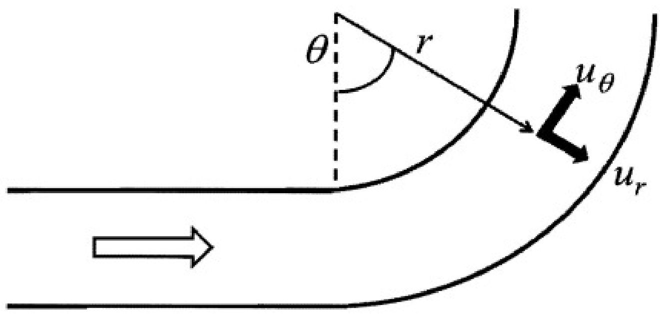

Fig. 1

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Vortex Dynamics Data Card

Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint \boldsymbol{A} \cdot d\boldsymbol{S}$

Stokes : $\int (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{S} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$

Vector Identities

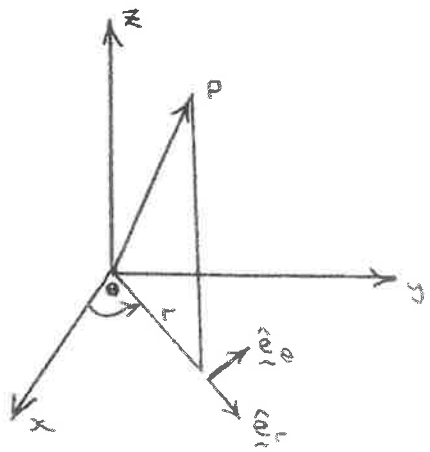
$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B}.\nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})$
$\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot \nabla f$
$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$
$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$

Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$



$$\nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \right)$$

Cambridge University Engineering Department

4A12: Turbulence

Data Card

Assume incompressible fluid with constant properties.

Continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

Mean momentum:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

Mean scalar:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D \frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

Turbulent kinetic energy ($k = \overline{u_i' u_i'}/2$):

$$\begin{aligned}\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} \\ &\quad - \overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'}\end{aligned}$$

The $k - \varepsilon$ model:

$$\begin{aligned}\frac{\partial k}{\partial t} + \overline{u}_i \frac{\partial k}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + P_k - \varepsilon \\ \frac{\partial \varepsilon}{\partial t} + \overline{u}_i \frac{\partial \varepsilon}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + c_{\varepsilon 1}\frac{\varepsilon}{k}P_k - c_{\varepsilon 2}\frac{\varepsilon^2}{k}\end{aligned}$$

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

$$P_k = \frac{1}{2}\nu_t \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)^2$$

$$C_\mu = 0.09, \; c_{\varepsilon 1} = 1.44, \; c_{\varepsilon 2} = 1.92, \; \sigma_k = 1.0, \; \sigma_\varepsilon = 1.3$$

Energy dissipation:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

Scalar fluctuations ($\sigma^2 = \overline{\phi'\phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D \frac{\partial^2 \sigma^2}{\partial x_j^2} - 2\overline{\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'} \frac{\partial \overline{\phi}}{\partial x_j} - 2D \overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

Scalar fluctuations (modelled):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_i \frac{\partial \sigma^2}{\partial x_i} = \frac{\partial}{\partial x_i} \left((D + D_{turb}) \frac{\partial \sigma^2}{\partial x_i} \right) + 2D_{turb} \left(\frac{\partial \overline{\phi}}{\partial x_i} \right)^2 - 2\overline{N}$$

Scalar dissipation:

$$2\overline{N} = 2D \overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2 = 2\frac{u}{L_{turb}}\sigma^2$$

Scaling rule for shear flow, flow dominant in direction x_1:

$$\frac{u}{L_{turb}} \sim \frac{\partial \overline{u}_1}{\partial x_2}$$

Kolmogorov scales:

$$\begin{aligned} \eta_K &= \left(\nu^3/\varepsilon\right)^{1/4} \\ \tau_K &= \left(\nu/\varepsilon\right)^{1/2} \\ v_K &= \left(\nu\varepsilon\right)^{1/4} \end{aligned}$$

Taylor microscale:

$$\varepsilon = 15\nu \frac{u^2}{\lambda^2}$$

Eddy viscosity (general):

$$\begin{aligned} \overline{u_i' u_j'} &= -\nu_{turb} \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) + \frac{2}{3} k \delta_{ij} \\ \overline{u_j' \phi'} &= -D_{turb} \frac{\partial \overline{\phi}}{\partial x_j} \end{aligned}$$

Eddy viscosity (for simple shear):

$$\overline{u_1' u_2'} = -\nu_{turb} \frac{\partial \overline{u}_1}{\partial x_2}$$