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Q1 (a) The mean scalar transport eqtn is

$$\frac{\partial \Phi}{\partial t} + U_1 \frac{\partial \Phi}{\partial x_1} + U_2 \frac{\partial \Phi}{\partial x_2} + U_3 \frac{\partial \Phi}{\partial x_3} = -\frac{\partial (\overline{u_1'\varphi'})}{\partial x_1} - \frac{\partial (\overline{u_2'\varphi'})}{\partial x_2} - \frac{\partial (\overline{u_3'\varphi'})}{\partial x_3}$$

(ignore molecular diff)

$U_2 = U_3 = 0$ (1-dim flow)

$\frac{\partial \Phi}{\partial x_1} = 0$, $\frac{\partial \Phi}{\partial x_3} = 0$ (homogeneity, given in the question)

$\frac{\partial \Phi}{\partial t} = 0$ (statistically-steady)

$\frac{\partial (\overline{u_1'\phi'})}{\partial x_1} = \frac{\partial (\overline{u_3'\varphi'})}{\partial x_3} = 0$ (homogeneous assumption)

$\Rightarrow 0 = -\frac{\partial}{\partial x_2} \overline{u_2'\varphi'} \iff \overline{u_2'\varphi'} =$ constant in x_2

(b) σ^2 eqtn (neglecting molecular diffusion) from the data card:

$$\underbrace{\frac{\partial \sigma^2}{\partial t}}_{\substack{=0 \\ \text{statistically-steady}}} + \underbrace{U_1 \frac{\partial \sigma^2}{\partial x_1} + U_2 \frac{\partial \sigma^2}{\partial x_2} + U_3 \frac{\partial \sigma^2}{\partial x_3}}_{\text{all } = 0 \text{ by the assumption in the question}} = \frac{\partial}{\partial x_i} \underbrace{\left(D_t \frac{\partial \sigma^2}{\partial x_i}\right)}_{=0 \text{ for all } i=1,2,3}$$

$$+ \underbrace{2 D_t \left(\frac{\partial \Phi}{\partial x_1}\right)^2}_{=0} + 2 D_t \left(\frac{\partial \Phi}{\partial x_2}\right)^2 + \underbrace{2 D_t \left(\frac{\partial \Phi}{\partial x_3}\right)^2}_{=0}$$

$$- 2 \cdot \frac{u_t}{L_t} \sigma^2 \quad \text{(modelled scalar dissipation)}$$

$$\Rightarrow 2 \frac{u_t}{L_t} \sigma^2 = 2 D_t \left(\frac{\partial \Phi}{\partial x_2}\right)^2$$

The r.h.s. is the production term $-2 \overline{u_2'\varphi'} \frac{\partial \Phi}{\partial x_2}$. We are told that $\overline{u_2'\varphi'} = \rho u_t \sigma$

$$\Rightarrow 2 \frac{u_t}{L_t} \sigma^2 = 2 \rho u_t \sigma \cdot S \quad \left(S = \frac{\partial \Phi}{\partial x_2}\right)$$

$$\Rightarrow \sigma = -\rho S L_t$$

Since $\sigma > 0$, ρ must be < 0.

(c) $D_t = C\, u_t L_t$

Going back to the modelled σ^2- eqtn,

$$2 \cdot C\, u_t L_t \left(\frac{\partial \Phi}{\partial x_2}\right)^2 = 2\, \frac{u_t}{L_t}\, \sigma^2$$

Using the result from part (b) that $\sigma = -\rho S L_t$, we get that

$$2 \cdot C \cdot u_t L_t \cdot S^2 = 2 \cdot \frac{u_t}{L_t} \cdot \rho^2 S^2 L_t^2$$

$\Leftrightarrow \quad C = \rho^2 \Rightarrow |\rho| = 0.32$. This is a quite reasonable value for the correlation coefficient between turbulent quantities.

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Q2 (a)

$Re_t = \dfrac{u L_t}{\nu}$ u: characteristic turbulent velocity

$\eta_k = (\nu^3/\varepsilon)^{1/4}$ If $\varepsilon = \dfrac{u^3}{L_t}$,

$\eta = (\nu^3 L_t / u^3)^{1/4} = (\nu^3/(u^3 L_t^3) \cdot L_t^4)^{1/4}$

$= L_t \left(\dfrac{1}{Re_t}\right)^{3/4}$ QED

(b) In the energy cascade, energy dissipation is the same across eddy range. In the inertial subrange, the dissipation is independent of scale and viscosity is not important. Therefore, the energy spectrum E must be a function only of the wavenumber k & the dissipation ε, $\Rightarrow E(k) = f(k, \varepsilon)$

Units: k : m^{-1}

ε : $m^2 s^{-3}$

E is defined such that $\underset{\substack{\uparrow \\ \text{turb kinetic} \\ \text{energy}}}{k} = \int_0^\infty E(\underset{\substack{\uparrow \\ \text{wavenumber}}}{k}) dk$

$\Rightarrow$ units of E : $m^3 s^{-2}$

We have 3 quantities, 2 units $\Rightarrow$

$E(k) = C\, k^{-5/3} \varepsilon^{2/3}$

(c) If ν is doubled & U is doubled, bulk flow Re stays the same. We assume that $\sqrt{k}/U$ = constant (i.e. turbulent velocity is same fraction of mean velocity when the mean velocity is doubled). The integral lengthscale stays the same because it depends on the size of the pipe $\Rightarrow L_t$ stay constant

$\Rightarrow$ turbulent Reynolds number stays const.

$\Rightarrow \eta_k = L_t Re_t^{-3/4}$ stays constant

$\varepsilon = u^3/L_t$ stays constant

$\dfrac{L_t}{u_t}$ is halved, $u_t^2 = k$ is $\times 4$, The spectrum E needs careful discussion. In terms of how it is spread along the wavenumber axis, since L_t & η_K stay the same, the $E(k)$ range does not change. Since the integral under the $E(k)$ curve is the turb kin energy, E will be $\times 4$ when the mean flow is doubled.

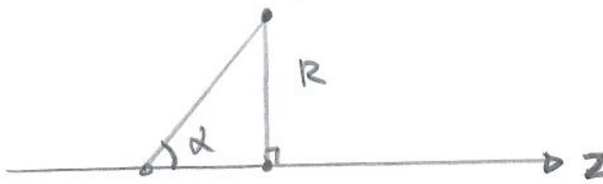
Q3 - 1

(a) Biot - Savart law $\underline{u}(\underline{x}) = \frac{1}{4\pi} \int \frac{\underline{w}(\underline{x}') \times (\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^{3/2}} dv'$

Γ : Strength of the vortex.

Without loss of generality we compute the velocity at the point P, a radial distance R from the origin.

The vortex induces a velocity u_θ



$$u_\theta = \frac{\Gamma}{4\pi} \int_{z=-\infty}^{z=+\infty} \frac{R\,dz}{(R^2+z^2)^{3/2}} = \frac{\Gamma}{4\pi R} \left(\frac{z}{(R^2+z^2)^{1/2}} \right)_{-\infty}^{+\infty} = \frac{\Gamma}{2\pi R}$$

(b) Consider an origin of coordinates at C

$$r_1 = \frac{\Gamma_2 L}{\Gamma_1 + \Gamma_2} \quad , \quad r_2 = \frac{\Gamma_1 L}{\Gamma_1 + \Gamma_2}$$

$$\boxed{\text{The velocity } \underline{u}_c = 0}$$

Relative to C, the angular velocity at A is $\Omega_A = \frac{u_y(A)}{r_1} =$

$$\frac{-\frac{\Gamma_2}{2\pi L}}{\frac{\Gamma_2 L}{\Gamma_1 + \Gamma_2}} = \frac{\Gamma_1 + \Gamma_2}{2\pi L^2}$$

the angular velocity at B is

$$\Omega_B = \frac{u_y(B)}{r_2} = \frac{\Gamma_1 / 2\pi L}{\frac{\Gamma_1 L}{\Gamma_1 + \Gamma_2}} = \frac{\Gamma_1 + \Gamma_2}{2\pi L^2}$$

(c) The velocities at $\underline{u}_A$, $\underline{u}_B$ & $\underline{u}_c$ are

$$\underline{u}_A = \left(0, -\frac{\Gamma_2}{2\pi L}\right), \quad \underline{u}_B = \left(0, \frac{\Gamma_1}{2\pi L}\right), \quad \underline{u}_c = (0, 0)$$

Q3 - 2

(d) (i) $\Gamma_1 = \Gamma_2$, $\underline{u}_c = 0$, $\Omega = \Gamma / \pi l^2$

(ii) $\Gamma_1 = \Gamma_2$, $\underline{u}_c = 0$, $\Omega = 0$

(i) Flow rotate at $\Omega = \Gamma / \pi l^2$, c doesn't move. the trailing vortices swept from the deck of an aircraft carrier is an example.

(ii) c is at infinity and doesn't move. the tip vortices of an aeroplane is an example.

(e) Initialy the strengths, radii and speed of the two vortices are equal, but the rear vortex lies in the velocity field of the first, and vice versa. The interaction of the two vortices consists of an enlargement of the more advanced vortex, and, simultaneously, in a dimnation of the rear vortex. The rear vortex then speed up, in accordance with $u = \frac{\Gamma}{4\pi R} \ln\left(\frac{R}{a}\right)$, and passes through the first. The situation is then reversed and the front vortex expands back to its original radius.

Q4-1

(a) $$\frac{\partial u_z}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(r u_r) = 2a \qquad \text{(From continuity)}$$

$$u_z = 2az$$

(b) From θ equation

$$\underbrace{-ar\frac{du_\theta}{dr} - a u_\theta}_{-a\frac{d}{dr}(r u_\theta)} = \nu \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r u_\theta)\right]$$

(c) $$f = \frac{2\pi r u_\theta}{\Gamma} \qquad -a\frac{df}{dr} = \nu\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right)$$

$$\boxed{\frac{d^2 f}{dr^2} + \left(\frac{ar}{\nu} - \frac{1}{r}\right)\frac{df}{dr} = 0}$$

(d) $$\frac{d}{dr}\left(\ln\frac{df}{dr}\right) = \frac{d^2 f}{dr^2} \Big/ \frac{df}{dr} = \frac{1}{r} - \frac{ar}{\nu}$$

$$\ln\frac{df}{dr} = \ln r - \frac{a}{2\nu}r^2 + A$$

$$\frac{df}{dr} = A r e^{-\left(\frac{a}{2\nu}r^2\right)}$$

$$f = A e^{-\frac{ar^2}{2\nu}} + B$$

$$f(0) = 0 \Rightarrow \quad f = B\left(1 - e^{-\frac{ar^2}{2\nu}}\right)$$

$$f(r\to\infty) = 1, \Rightarrow B = 1, \quad f = \left(1 - e^{-\frac{ar^2}{2\nu}}\right)$$

$$\boxed{u_\theta = \frac{\Gamma}{2\pi r}\left(1 - e^{-\frac{ar^2}{2\nu}}\right)}$$

This is the Burgers' Vortex. Outward diffusion is balanced by the inward flow + vortex stretching.