EGT3
ENGINEERING TRIPOS PART IIB

Monday 5 May 2025 9.30 to 11.10

Module 4A12

TURBULENCE AND VORTEX DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
The NS equations in the Cylindrical Coordinates and the Biot-Savart law (1 page)
4A12 Data Card (3 pages)
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 Consider a statistically-steady homogeneous isotropic non-decaying turbulent flow with mean velocity U in the streamwise direction x_1, and zero mean velocity in the other two directions (x_2 and x_3). The turbulence has constant characteristic velocity fluctuation u_t and integral lengthscale L_t. There exists a constant scalar gradient in the x_2 direction so that $\partial\Phi/\partial x_2 = S$, where Φ is the mean of a scalar ϕ and S is a positive constant. All first and higher moments of all turbulence quantities are homogeneous in the x_1 and x_3 directions. σ is the characteristic scalar fluctuation ($\sigma^2 = \overline{\phi'\phi'}$) and is taken as homogeneous in all three directions. Ignore molecular diffusion.

(a) By reference to the mean scalar transport equation and not otherwise, show that the turbulent flux $\overline{u_2'\phi'}$ is constant. [30%]

(b) The correlation coefficient ρ between the velocity and the scalar fluctuations is constant, so that $\overline{u_2'\phi'} = \rho u_t \sigma$. By considering the balance of terms in the σ^2 transport equation and not otherwise, show that $\sigma = -\rho S L_t$, stating all assumptions made. What does this imply for the sign of ρ? [40%]

(c) Now consider the usual eddy viscosity model for $\overline{u_2'\phi'}$, with the eddy viscosity given by $C u_t L_t$ and where C is a constant. Consider again the σ^2 transport equation and the result of part (b). If an experiment gives that $C = 0.1$, what is the numerical value of ρ? Is this estimate reasonable? [30%]

2 (a) Show that $\eta_K = L_t Re_t^{-3/4}$, where η_K is the Kolmogorov lengthscale, L_t the integral lengthscale, and Re_t the turbulent Reynolds number. [30%]

(b) List the properties of the *inertial sub-range* of the energy spectrum E and derive an expression for the wavenumber dependence of E in this sub-range. [40%]

(c) Discuss what happens to the integral lengthscale, the eddy turnover time, the turbulent kinetic energy, the Kolmogorov lengthscale, and the energy spectrum at a particular point in a turbulent pipe flow if the flow rate is doubled and the kinematic viscosity of the fluid is also doubled. State clearly your assumptions and explain your reasoning. [30%]

 (TURN OVER

3 (a) Use the Biot-Savart law to calculate the velocity field of a line vortex with strength Γ. [20%]

(b) Suppose there are two point vortices at A and B of strengths Γ_1 and Γ_2 in an inviscid fluid as shown in Fig. 1 (a). The distance between A and B is l.

(i) Find the location of a point C on line AB, such that the motion of points A and B, relative to C, is pure rotation with the same angular velocity Ω. What is the value of Ω? [30%]

(ii) What are the velocities $\mathbf{u}_A$, $\mathbf{u}_B$, and $\mathbf{u}_C$ at A, B and C? [10%]

(iii) Consider two special cases: (i) $\Gamma_1 = \Gamma_2$ and (ii) $\Gamma_1 = -\Gamma_2$. What are the values of $\mathbf{u}_C$ and Ω? Give an interpretation of these values. Give an example of a real flow to illustrate each case. [20%]

(c) Two vortex rings with the same strengths, radii and speeds are initially placed one behind the other as shown in Fig. 1 (b). Describe and explain the motion of the two vortex rings. [20%]

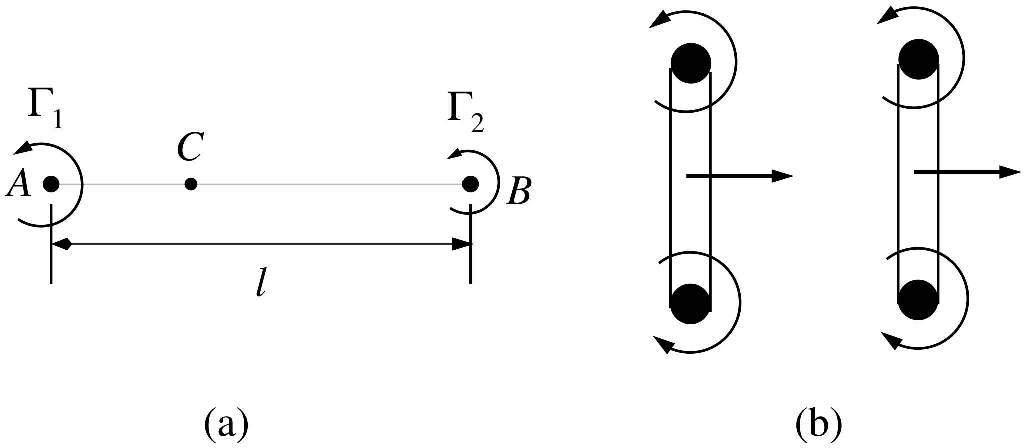


Fig. 1

4 Consider a steady axisymmetric flow with velocity field (u_r, u_θ, u_z). A symmetric radial inflow is defined as:

$$u_r = -ar$$

where a is a constant.

(a) Assume that $u_z = 0$ at $z = 0$. Determine the velocity component u_z. [10%]

(b) If u_θ depends only on r, derive an ordinary differential equation for u_θ from the Navier-Stokes equations in cylindrical coordinates. Use the expressions for u_r and u_z from above. [20%]

(c) A new dependent variable is defined as

$$f = \frac{2\pi r u_\theta}{\Gamma}$$

Derive the equation for f from that of u_θ. [30%]

(d) The boundary conditions are

$$u_\theta(r=0) = 0 \text{ and } u_\theta(r \to \infty) = \frac{\Gamma}{2\pi r}$$

Solve the equation for f and hence u_θ. Interpret your results. [40%]

END OF PAPER

THIS PAGE IS BLANK

Navier-Stokes Equations in the Cylindrical Coordinates

$$\text{Continuity:} \quad \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta} + \frac{\partial(u_z)}{\partial z} = 0 \ ,$$

$$r: \quad \rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right],$$

$$\theta: \quad \rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right] \ ,$$

$$z: \quad \rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right] \ .$$

The Biot-Savart Law

$$\mathbf{u}(\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{4}\pi}\int \frac{\omega(\mathbf{x'}) \times (\mathbf{x} - \mathbf{x'})\mathbf{dV'}}{|\mathbf{x} - \mathbf{x'}|^{\mathbf{3}}}$$

.

Vortex Dynamics Data Card

Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint \boldsymbol{A} \cdot d\boldsymbol{S}$

Stokes : $\int (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{S} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$

Vector Identities

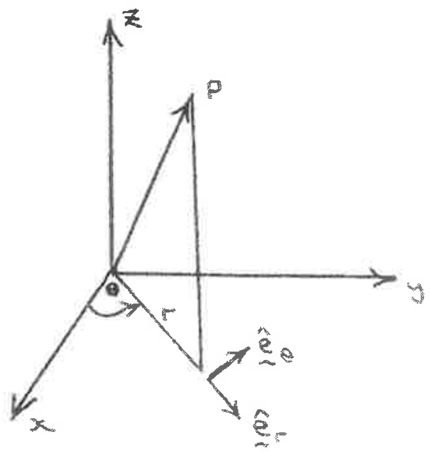
$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B}.\nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})$
$\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot \nabla f$
$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$
$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$

Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \frac{1}{r} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$



$$\nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \right)$$

Cambridge University Engineering Department

4A12: Turbulence

Data Card

Assume incompressible fluid with constant properties.

Continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

Mean momentum:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

Mean scalar:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D \frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

Turbulent kinetic energy ($k = \overline{u_i' u_i'}/2$):

$$\begin{aligned} \frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} \\ &\quad - \overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'} \end{aligned}$$

The $k - \varepsilon$ model:

$$\frac{\partial k}{\partial t} + \overline{u}_i \frac{\partial k}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + P_k - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + \overline{u}_i \frac{\partial \varepsilon}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + c_{\varepsilon 1}\frac{\varepsilon}{k}P_k - c_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

$$P_k = \frac{1}{2}\nu_t \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)^2$$

$$C_\mu = 0.09, \;\; c_{\varepsilon 1} = 1.44, \;\; c_{\varepsilon 2} = 1.92, \;\; \sigma_k = 1.0, \;\; \sigma_\varepsilon = 1.3$$

Energy dissipation:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

Scalar fluctuations ($\sigma^2 = \overline{\phi'\phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D\frac{\partial^2 \sigma^2}{\partial x_j^2} - 2\overline{\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'}\frac{\partial \overline{\phi}}{\partial x_j} - 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

Scalar fluctuations (modelled):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_i \frac{\partial \sigma^2}{\partial x_i} = \frac{\partial}{\partial x_i}\left((D + D_{turb})\frac{\partial \sigma^2}{\partial x_i}\right) + 2D_{turb}\left(\frac{\partial \overline{\phi}}{\partial x_i}\right)^2 - 2\overline{N}$$

Scalar dissipation:

$$2\overline{N} = 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2 = 2\frac{u}{L_{turb}}\sigma^2$$

Scaling rule for shear flow, flow dominant in direction x_1:

$$\frac{u}{L_{turb}} \sim \frac{\partial \overline{u}_1}{\partial x_2}$$

Kolmogorov scales:

$$\begin{aligned}
\eta_K &= \left(\nu^3/\varepsilon\right)^{1/4} \\
\tau_K &= \left(\nu/\varepsilon\right)^{1/2} \\
v_K &= \left(\nu\varepsilon\right)^{1/4}
\end{aligned}$$

Taylor microscale:

$$\varepsilon = 15\nu\frac{u^2}{\lambda^2}$$

Eddy viscosity (general):

$$\begin{aligned}
\overline{u_i' u_j'} &= -\nu_{turb}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij} \\
\overline{u_j' \phi'} &= -D_{turb}\frac{\partial \overline{\phi}}{\partial x_j}
\end{aligned}$$

Eddy viscosity (for simple shear):

$$\overline{u_1' u_2'} = -\nu_{turb}\frac{\partial \overline{u}_1}{\partial x_2}$$