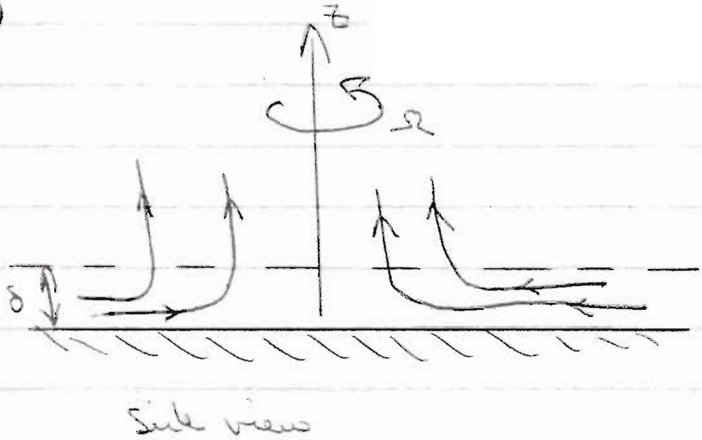
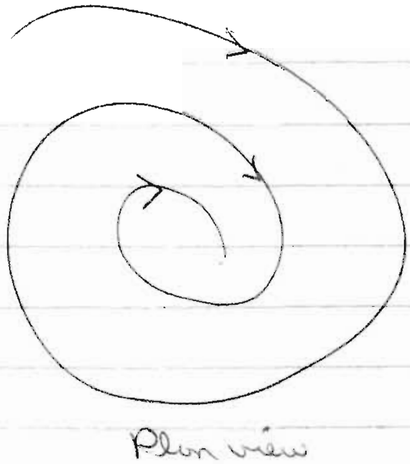
1 (a)





Outside Bodewadt layer : $\frac{dp}{dr} = \rho \frac{u_\theta^2}{r} = \rho \Omega^2 r$

This pressure gradient is imposed on fluid in the Bodewadt layer, creating a radial pressure force.

In the Bodewadt layer $u_\theta^2/r < \Omega^2 r$, and so the radial pressure force exceeds the local centripetal acceleration. The excess pressure force drives the inflow

$\delta = f(\nu, \Omega)$ [No other variables]

δ: m, ν: m²/s, Ω: s⁻¹

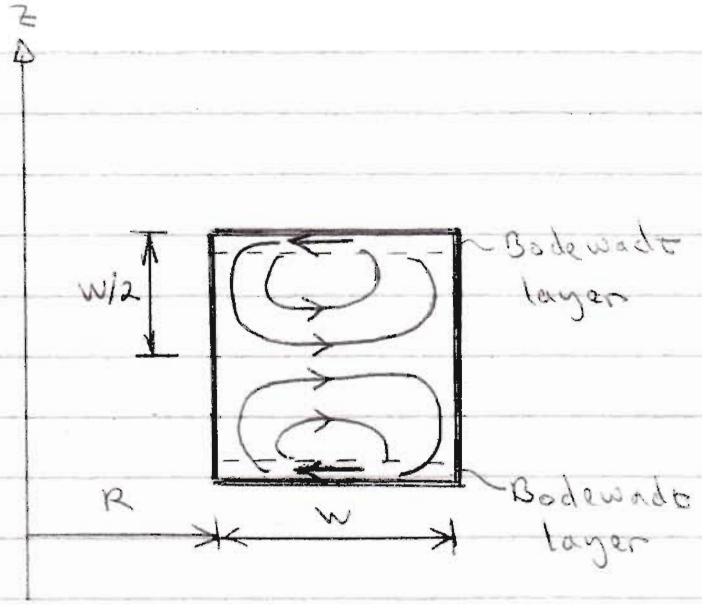
Π Theorem: variables = 3, dimensions = 2 ⇒ groups = 1

Only 1 dimensionless group : $\frac{\delta^2 \Omega}{\nu}$ (by inspection)

No other dimensionless groups ⇒ $\frac{\delta^2 \Omega}{\nu}$ = constant

$$\Rightarrow \underline{\underline{\delta \sim \sqrt{\nu/\Omega}}}$$

(b)



In Bodewadt layer $|u_r| \sim u_\theta \sim \Omega r$

Continuity $\Rightarrow$ $u_{core} W/2 \sim |u_r| \delta$, $(\delta \sim \sqrt{\nu/\Omega})$

$\Rightarrow$ $u_{core} W/2 \sim \Omega \sqrt{\nu/\Omega}\,(R + W/2)$

$$\Rightarrow \quad u_{core} \sim \frac{\Omega(2R+W)\sqrt{\nu/\Omega}}{W} = \underline{\underline{\frac{(2R+W)\sqrt{\nu\Omega}}{W}}}$$

(c) The Ekman pumping flushes all the fluid through the Bodewadt layer, which is very dissipative. As each fluid element passes through the Bodewadt layer a significant % of its kinetic energy is destroyed. The rate of working of the pressure forces replenishes this KE.

$\Rightarrow$ $\Delta p\,(\dot{m}/\rho)$ = rate of destruction of energy in Bodewadt layer

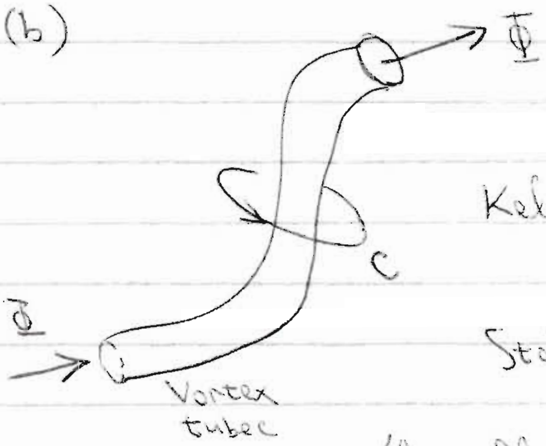
Thus Δp controlled by Ekman pumping.

2 (a) Helmholtz's laws apply only to inviscid flows. They are:

1. The vortex lines are frozen into the fluid, like dye lines.

2. The flux of vorticity, $\Phi = \int \underset{\sim}{\omega} \cdot d\underset{\sim}{A}$, is constant along a vortex tube and independant of time.

(b)



$$\Phi = \int \underset{\sim}{\omega} \cdot d\underset{\sim}{A}$$

Kelvin: $\oint_C \underset{\sim}{u} \cdot d\underset{\sim}{r} = \text{constant}.$

Stokes: $\Phi = \int \underset{\sim}{\omega} \cdot d\underset{\sim}{A} = \oint \underset{\sim}{u} \cdot d\underset{\sim}{r}$.

Thus flux of vorticity through C is independent of time.

$\Rightarrow$ C must encircle vortex tube for all t

This is true of all material curves, C, that encircle the flux tube at $t=0$. This is possible only if the vortex tube moves with the fluid.

(c) $$\frac{D\underset{\sim}{B}}{Dt} = \frac{D}{Dt} d\underset{\sim}{r} - \alpha \frac{D\underset{\sim}{\omega}}{Dt} \quad \text{(since } \alpha \text{ is a constant)}$$

$$= (d\underset{\sim}{r} \cdot \nabla)\underset{\sim}{u} - \alpha \frac{D\underset{\sim}{\omega}}{Dt}$$

In an inviscid fluid, $\frac{D\underset{\sim}{\omega}}{Dt} = (\underset{\sim}{\omega} \cdot \nabla)\underset{\sim}{u}$

$$\Rightarrow \frac{D\underset{\sim}{B}}{Dt} = (d\underset{\sim}{r} \cdot \nabla)\underset{\sim}{u} - \alpha(\underset{\sim}{\omega} \cdot \nabla)\underset{\sim}{u} = (\underset{\sim}{B} \cdot \nabla)\underset{\sim}{u}$$

At $t=0$, $\underset{\sim}{B}=0$ by virtue of the definition of α.

$$\Rightarrow \quad \frac{D\underset{\sim}{B}}{Dt} = (\underset{\sim}{B}\cdot\nabla)\underset{\sim}{u} = 0 \quad \text{at } t=0$$

$$\Rightarrow \quad \underset{\sim}{B}=0 \quad \text{a short time later.}$$

$$\Rightarrow \quad \underset{\sim}{B}=0 \quad \text{for all time (by itteration)}$$

If $\underset{\sim}{B}=0$ for all t,

$$d\underset{\sim}{r} = \alpha\,\underset{\sim}{\omega}(\underset{\sim}{x}_A, t) \quad \text{for all time.}$$

Thus $\underset{\sim}{\omega}(\underset{\sim}{x}_A, t)$ is always $//^{\text{el}}$ to $d\underset{\sim}{r}$, and so $\underset{\sim}{\omega}$ moves like a dye line. So A and B always lie on the vortex line.

3 (a) From lecture notes or textbooks:

Energy cascade is the concept by which we imply the transfer of turbulent kinetic energy from the large eddies to the smaller eddies, eventually to be dissipated by viscosity to heat.

The key assumption is that there is no molecular dissipation of energy as the kinetic energy flows down the cascade, which means that for any scale r,

$$\varepsilon = \frac{u^3}{L_t} = \frac{v(r)^3}{r} = \frac{v_K^3}{\eta_K}$$

u: large scale vel fluctuations

L_t: integral length scale

v_K: Kolmogorov velocity

η_K: " " length

ε: dissipation, independent of scale from L_t to η_K.

The inertial range of the turbulent spectrum exists at high Reynolds numbers. It refers to the range of scales between the flow-dependent large energy-containing scales and the Kolmogorov scale where viscosity is important.

(b) If $E(\lambda)$ = function of ε & λ only

(the key assumption of inertial range universal Kolmogorov theory), and since units of $E(\lambda)$ are $\left[\frac{m^3}{s^2}\right]$

(recall: $\int_0^\infty E(\lambda)\,d\lambda = u^2$), unit of λ is $[m^{-1}]$ and

units of ε are $\left[\frac{m^2}{s^3}\right]$, we have 3 quantities

and 2 units and by Buckingham's Π-theorem

we have 1 dimensionless group only

3. (b) cont'd

$$\Rightarrow \quad \frac{E(\lambda)}{\varepsilon^{p}\,\lambda^{q}} = \text{Constant}$$

$$\therefore \quad m^{3}\,s^{-2} = m^{-q}\; m^{2p}\; s^{-3p}$$

$$\Rightarrow \begin{cases} -2 = -3p \Rightarrow p = 2/3 \\ 3 = -q + 2p \Rightarrow 3 = -q + \frac{4}{3} \Rightarrow q = -5/3 \end{cases}$$

$$\therefore \quad \underline{\underline{E(\lambda) = C\;\varepsilon^{2/3}\;\lambda^{-5/3}}}$$

(c) The spectrum of the temperature fluctuations $E_\theta(\lambda)$ is such that $\int_0^\infty E_\theta(\lambda)\,d\lambda = \theta^2$, θ: characteristic temperature energy of the fluctuations, $\Rightarrow$ units of $E_\theta(\lambda) : [K^2\,m]$

The scalar dissipation $N \sim \frac{\theta^2}{T_t}$ T_t: eddy turnover time

$$\Rightarrow \text{units of } N : [K^2\,s^{-1}]$$

If $E_\theta(\lambda) = f(N, \varepsilon, \lambda)$ $\Rightarrow$ 4 quantities, 3 units $\Rightarrow$ 1 non-dim group

$$\Rightarrow \quad E_\theta(\lambda) = C_\theta\, N^{p}\,\varepsilon^{q}\,\lambda^{r}$$

$$K^2\,m = (K^2\,s^{-1})^{p}\,(m^2\,s^{-3})^{q}\,(m^{-1})^{r}$$

$$\Rightarrow \quad p = 1, \quad q = -1/3, \quad r = -5/3$$

$$\Rightarrow \boxed{E_\theta(\lambda) = C_\theta\,N\,\varepsilon^{-1/3}\,\lambda^{-5/3}}$$

Tocuman

4.(a) k-eqtn:

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2} \frac{\partial \overline{u_j u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2}$$

$$- \overline{u_i' u_j'} \frac{\partial \bar{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'}$$

From LHS : T1 = unsteady $o\left(\frac{u^2}{T}\right)$, T: timescale of mean flow change if not statistically stationary

T2 : advection $o\left(\bar{U} \frac{u^2}{L}\right)$

$\bar{U}$: mean flow; L: width of flow (L_t is OK)

u^2: characteristic turb velocity squared ($\sqrt{k}$ is also OK)

L_t: integral lengthscale

RHS T1 : pressure gradient work, transports k in the fluid $= o\left(\frac{\rho u^2}{\rho L_t} u\right) = o\left(\frac{u^3}{L_t}\right)$

T2 : turbulent diffusion, $o\left(\frac{u^3}{L_t}\right)$

T3 : molecular diffusion of k, $o\left(\nu \frac{u^2}{L_t^2} = \frac{\nu}{u L_t} \frac{u^3}{L_t}\right)$

$= o\left(\frac{1}{Re_t} \frac{u^3}{L_t}\right)$

i.e. negligible at high Re_t

T4 : production by mean shear

$o\left(u^2 \frac{\bar{U}}{L}\right) = o\left(\frac{u^3}{L_t}\right)$ since $o\left(\frac{\bar{U}}{L}\right) = o\left(\frac{u}{L_t}\right)$

T5 : Dissipation of kinetic energy ε

order of magnitude $= \frac{u^3}{L_t}$

or $o\left(\nu \frac{u^2}{\lambda^2}\right)$, λ = Taylor microscale

T6 : Body force. Order of magnitude depends on problem.

4(b). k-ε model for homogeneous flows (i.e. all spatial gradients of mean quantities $=0$)

$$\frac{dk}{dt} = -\varepsilon \qquad (1)$$

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon 2}\frac{\varepsilon^2}{k} \qquad (2)$$

If $k = c_1 t^{-p} \Rightarrow \dfrac{dk}{dt} = (-p)\, c_1 t^{-p-1}$

If $\varepsilon = c_2 t^{-q} \Rightarrow \dfrac{d\varepsilon}{dt} = (-q)\, c_2 t^{-q-1}$

Put in (1): $(-p)\, c_1 t^{-p-1} = -C_{\varepsilon 2} t^{-q}$

$\Leftrightarrow p c_1 t^{-p-1} = C_{\varepsilon 2} t^{-q} \Rightarrow 1+p = q$

since this must be so for all times

Put in (2): $(-q) c_2 t^{-q-1} = -C_{\varepsilon 2}\dfrac{c_2^2 t^{-2q}}{c_1 t^{-p}}$

$\Leftrightarrow -q-1 = -2q+p \Leftrightarrow \underline{q = 1+p}$

Note: we arrive at the same dependence. This is no coincidence, the modelled ε equation has been designed to bring this consistency

Note also that by performing experiments and building knowledge on p, q, c_1, c_2 one gets some of the constants in the k-ε model

Let's go back:

$L_t = \dfrac{k^{3/2}}{\varepsilon} \Rightarrow L_t = \dfrac{c_1^{3/2} t^{-3/2 p}}{c_2 t^{-q}} = \dfrac{c_1^{3/2}}{c_2} t^{1-p/2}$ (using result above)

$\Rightarrow \underline{\underline{L_t \sim t^{3/8}}}$ for $p = 5/4$

$T_t = \dfrac{k}{\varepsilon} \Rightarrow T = \dfrac{c_1 t^{-p}}{c_2 t^{-q}} = \dfrac{c_1}{c_2} t^{q-p}$

$\Rightarrow \underline{\underline{T_t \sim t}}$ (for any value of p)