4A12 Examiner's comments:

Question 1: The stretching and diffusion of vortex tubes
All students attempted this question and the overall performance was good. Most mistakes occurred in part c(i), which was quite challenging.

Question 2: Helmholtz's laws and Kelvin's Theorem
All students attempted this question and most answered it well. Some students had problems with the proof in part (b).

Question 3: The decay of turbulence
Part (a) of this question needed a deep understanding of turbulent kinetic energy balances to answer correctly, while part (b) was straightforward book-keeping. Part (c) was answered relatively poorly as few recognised that production is limited to close to the surface.

Question 4: Self-preservation applied to thin shear layers
A relatively straightforward question largely based on the lecture notes. However, the parts that needed critical thought were not answered well in general.

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1 (a) $\underset{\sim}{\omega} = \nabla \times \underset{\sim}{u}$ and axisymmetric

Data sheet gives $\omega_z = \dfrac{1}{r}\dfrac{\partial}{\partial r}(r u_\theta)$

$$\Rightarrow \quad r u_\theta = \int_0^r r\omega_z \, dr = \int_0^r \frac{\Phi_0}{\pi \delta^2} e^{-r^2/\delta^2} r\,dr$$

$$= \frac{\Phi_0}{2\pi}\int_0^r e^{-r^2/\delta^2}\, d\left(\frac{r^2}{\delta^2}\right)$$

Φ_0 = circulation around vortex
= flux of ω_z along vortex

$$= \frac{\Phi_0}{2\pi}\left[-e^{-r^2/\delta^2}\right]_0^r$$

$$\Rightarrow \quad u_\theta = \frac{\Phi_0}{2\pi r}\left[1 - \exp(-r^2/\delta^2)\right]$$

(b) $$\frac{D\omega_z}{Dt} = \frac{\partial \omega_z}{\partial t} + \underset{\sim}{u}\cdot\nabla\omega_z = \frac{\partial\omega_z}{\partial t} \qquad (\underset{\sim}{u}\cdot\nabla\omega = 0)$$

$$\Rightarrow \quad \frac{D\omega_z}{Dt} = -\frac{2}{\pi\delta^3}\frac{d\delta}{dt}\Phi_0 e^{-r^2/\delta^2} + \frac{\Phi_0}{\pi\delta^2}e^{-r^2/\delta^2}\left[\frac{2r^2}{\delta^3}\frac{d\delta}{dt}\right]$$

$$= -\frac{2\Phi_0}{\pi\delta^3}\frac{d\delta}{dt}e^{-r^2/\delta^2}\left[1 - \frac{r^2}{\delta^2}\right]$$

Also,
$$\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\omega_z}{\partial r}\right) = \frac{\nu\Phi_0}{\pi\delta^2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}e^{-r^2/\delta^2}\right)$$

$$= \frac{\nu\Phi_0}{\pi\delta^2}\frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{2r^2}{\delta^2}e^{-r^2/\delta^2}\right)$$

$$= -\frac{\nu 2\Phi_0}{\pi\delta^4}\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 e^{-r^2/\delta^2}\right)$$

$$= -\frac{\nu 2\Phi_0}{\pi\delta^3}\left[2e^{-r^2/\delta^3} - \frac{2r^2}{\delta^2}e^{-r^2/\delta^2}\right]$$

$$= -\frac{\nu 4\Phi_0}{\pi\delta^4}e^{-r^2/\delta^2}\left[1 - \frac{r^2}{\delta^2}\right]$$

Equate:
$$-\frac{2\Phi_0}{\pi\delta^3}\frac{d\delta}{dt}e^{-r^2/\delta^2}\left[1 - \frac{r^2}{\delta^2}\right] = -\frac{4\nu\Phi_0}{\pi\delta^4}e^{-r^2/\delta^2}\left[1 - \frac{r^2}{\delta^2}\right]$$

Cancellation:

$$\frac{d\delta}{dt} = \frac{2\nu}{\delta} \Rightarrow \frac{d\delta^2}{dt} = 4\nu \Rightarrow \delta = \delta_0 + \sqrt{4\nu t}$$

The $\delta \sim \sqrt{\nu t}$ growth is because of diffusion. Φ_0 is a constant as the total flux of ω_z cannot be altered by diffusion.

(c) New terms are: $\underset{\sim}{u} \cdot \nabla \omega_z$ on the left and $\omega_z \frac{\partial u_z}{\partial z}$ on right.

(i) $$\underset{\sim}{u} \cdot \nabla \omega_z = u_r \frac{\partial \omega_z}{\partial r} = \frac{1}{2} \alpha r \frac{2r}{\delta^2} \frac{\Phi_0}{\pi \delta^2} e^{-r^2/\delta^2} = \frac{\alpha r^2}{\delta^2} \frac{\Phi_0}{\pi \delta^2} e^{-r^2/\delta^2}$$

$$\omega_z \frac{\partial u_z}{\partial z} = \alpha \frac{\Phi_0}{\pi \delta^2} e^{-r^2/\delta^2}$$

$$\Rightarrow \underset{\sim}{u} \cdot \nabla \omega_z - \omega_z \frac{\partial u_z}{\partial z} = -\alpha \frac{\Phi_0}{\pi \delta^2} e^{-r^2/\delta^2} \left[1 - \frac{r^2}{\delta^2}\right]$$

Add to terms found in (b)

$$-\frac{2\Phi_0}{\pi \delta^3} \frac{d\delta}{dt} e^{-r^2/\delta^2} \left[1 - \frac{r^2}{\delta^2}\right] - \alpha \frac{\Phi_0}{\pi \delta^2} e^{-r^2/\delta^2} \left[1 - \frac{r^2}{\delta^2}\right] = \frac{-4\nu \Phi_0}{\pi \delta^4} e^{-r^2/\delta^2} \left[1 - \frac{r^2}{\delta^2}\right]$$

Cancellation: $$2\delta \frac{d\delta}{dt} + \alpha \delta^2 = 4\nu \Rightarrow \frac{d\delta^2}{dt} + \alpha \delta^2 = 4\nu$$

(ii) In the steady state there is a balance between:

- outward diffusion of ω_z and inward advection
- intensification of ω_z by vortex stretching and a reduction in ω_z by diffusion

(iii) Burgers' vortex has the property that, for $Re \gg 1$, the rate of dissipation of energy is independent of ν, which is also a property of turbulence. The small dissipative scales in turbulence resemble unsteady Burgers vortices.

2 (a) <u>Kelvin's theorem</u>: $\Gamma = \oint_C \underset{\sim}{u} \cdot d\underset{\sim}{l}$ = constant if C is a material curve.

<u>Helmholtz 1</u>: Vortex lines frozen into the fluid, like dye lines

<u>Helmholtz 2</u>: For a vortex tube, $\Phi = \int \underset{\sim}{\omega} \cdot d\underset{\sim}{S}$ is

(i) the same at all cross-sections of the tube

(ii) ~~the~~ independent of time

These laws apply only to inviscid fluids free of rotational body forces.

(b) $$\underbrace{\frac{D}{Dt}\oint_C}_{\text{commute}} \underset{\sim}{u}\cdot d\underset{\sim}{r} = \oint \frac{D\underset{\sim}{u}}{Dt}\cdot d\underset{\sim}{r} + \oint \underset{\sim}{u}\cdot \frac{D}{Dt} d\underset{\sim}{r} \quad \text{(chain rule)}$$

$$= \underbrace{\oint \left[-\nabla\left(\frac{p}{\rho}\right)\right]\cdot d\underset{\sim}{r}}_{\text{(Euler)}} + \underbrace{\oint \underset{\sim}{u}\cdot\left[d\underset{\sim}{r}\cdot\nabla\underset{\sim}{u}\right]}_{\text{(given result)}}$$

But $\underset{\sim}{u}\cdot(d\underset{\sim}{r}\cdot\nabla\underset{\sim}{u}) = (d\underset{\sim}{r}\cdot\nabla)(\frac{1}{2}\underset{\sim}{u}^2)$

$$\Rightarrow \frac{D}{Dt}\oint_C \underset{\sim}{u}\cdot d\underset{\sim}{r} = \underline{\underline{\oint \nabla\left[\tfrac{1}{2}\underset{\sim}{u}^2 - p/\rho\right]\cdot d\underset{\sim}{r} = 0}}$$

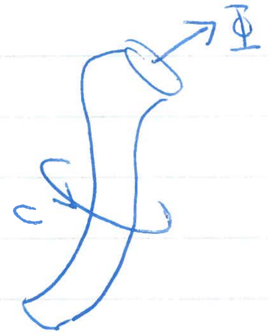
($\oint(\nabla f)\cdot d\underset{\sim}{r} = 0$ for any single valued function f)

(c) For inviscid fluid

$$\left.\begin{aligned} \frac{D\underset{\sim}{\omega}}{Dt} &= (\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{u} \\ \text{c.f.} \quad \frac{D}{Dt} d\underset{\sim}{r} &= (d\underset{\sim}{r}\cdot\nabla)\underset{\sim}{u} \end{aligned}\right\} \text{ same equation, so } \underset{\sim}{\omega} \text{ and } d\underset{\sim}{r} \text{ evolve in the same way}$$

If $\underset{\sim}{\omega}$ and $d\underset{\sim}{r}$ are coincident at $t=0$, the must remain coincident $\Rightarrow$ <u>vortex lines move like dye lines</u>

(d)



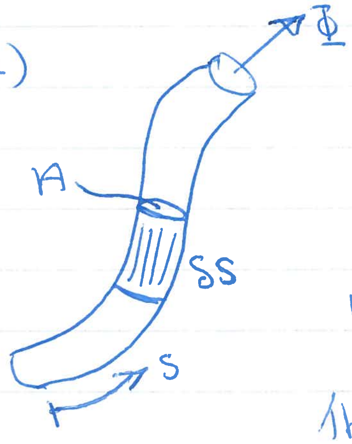
- Since $\underset{\sim}{\omega} = \nabla \times \underset{\sim}{u}$, $\nabla \cdot \underset{\sim}{\omega} = 0$ as $\nabla \cdot \nabla \times (\sim) \equiv 0$. Thus $\oint_S \underset{\sim}{\omega} \cdot d\underset{\sim}{S} = 0$ for any surface S. Since there is no flux out the sides of the vortex tube, Φ must be the same at all cross-sections.

- C is a material curve, and Helmholtz 1 tells us vortex lines move like dye lines, so C must continue to ~~[illegible]~~ encircle the tube for all times. But Kelvin's theorem tell us the flux of vorticity through C is a constant because Stokes' theorem gives us

$$\Gamma = \oint_C \underset{\sim}{u} \cdot d\underset{\sim}{r} = \int \underset{\sim}{\omega} \cdot d\underset{\sim}{S} = \Phi .$$

Thus the flux along the vortex tube is also conserved.

(e)



$\delta V = A(S)\, \delta S$ = constant as vortex lines frozen into fluid (Helmholtz 1)

Helmholtz 2 $\Rightarrow$ $\Phi = |\underset{\sim}{\omega}|\, A(S)$ = constant

Thus $\dfrac{\Phi}{\delta V} = \dfrac{\cancel{A}\, |\underset{\sim}{\omega}|}{\cancel{A}\, \delta S}$ = constant

$\Rightarrow$ ~~δS~~ $\boxed{|\underset{\sim}{\omega}| = \text{constant} \times \delta S}$

In turbulence, material lines are constantly stretched

$\Rightarrow$ on average, vortex lines are stretched

$\Rightarrow$ δS increases (on average)

$\Rightarrow$ $|\underset{\sim}{\omega}|$ amplified (untill diffusion cuts in)

Q3

(a) We assume stationary homogenous isotropic turbulence, with no production mechanism other than the distributed body force due to the cars' drag. Each car experiences a drag force equal to $F = \frac{1}{2}\rho V^2 C_d A$, where A is the frontal area of the vehicle. *NFV* is the total power injected to the air from all traffic, which means that the total kinetic energy per unit time injected to the city's air per unit mass is $P_{in} = (N\frac{1}{2}C_d A\rho V^2 V)/(\rho HLW)$. Since the turbulence is stationary, the turbulent kinetic energy dissipation must balance P_{in}, and with the usual model for dissipation, we get that $\varepsilon = P_{in} \Leftrightarrow \frac{u^3}{H/5} = \frac{NC_d A\rho V^2 V}{2\rho HLW}$. This gives that the characteristic turbulent velocity fluctuations are given by $u = \left(\frac{NC_d A}{10LW}\right)^{1/3} V$. (Note: as always in turbulence, it is interesting to do an order of magnitude check on whether this estimate may be realistic or not. Say that L=W=5 km, N=1000, C_d=0.3, A=2m^2, numbers that look reasonable for city traffic. This gives that u~0.013V. This is perhaps too low; we should expect something closer to V, given what we know about the turbulence in thin shear flows, thinking in terms of the wake behind each car. The reason is that we are distributing this turbulence across the whole of the city's air, i.e. across the whole height H. This is what is discussed in Part c of this question.)

(b) If traffic suddenly stops, we have decaying turbulence that obeys $\frac{dk}{dt} = -\varepsilon \Leftrightarrow \frac{dk}{dt} = -\frac{k^{3/2}}{L}$, where L is the turbulent lengthscale, assumed constant. This can be easily integrated to give k as a function of time. The result is $2L\left(k^{1/2} - k_0^{1/2}\right) = t$, with k_0 the initial kinetic energy, which can be expressed as $\frac{k}{k_0} = (1 + t/T_0)^{-2}$, where $T_0 = L/\sqrt{k_0}$ is the initial eddy turnover time.

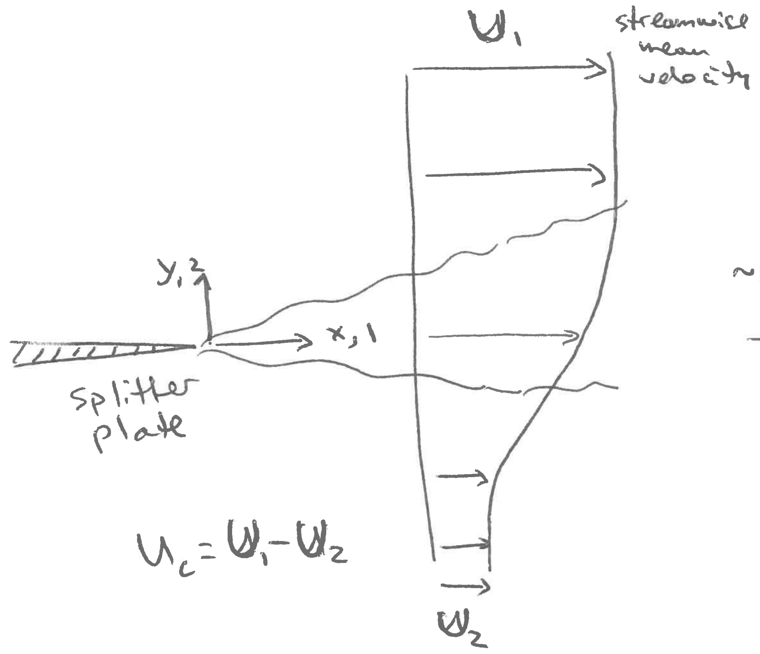
(c) The assumption that the lengthscale is constant needs discussion. As can be seen in the k-ε model (in the Data Card), the equation for the dissipation also includes k: in homogeneous turbulence, the ε-equation becomes $\frac{d\varepsilon}{dt} = -1.92\frac{\varepsilon^2}{k}$. This means that the lengthscale (equal to $k^{3/2}/\varepsilon$) is changing during the decay, in principle. To find out how exactly, we need a simultaneous solution of k and ε, which cannot be done easily analytically. Measurements of the decay rate of k and of the changes of L with time have led to the value of the empirical constant 1.92 in the k-ε model. Therefore, since dissipation will be decaying during the decay, the rate of change of k will be different than our analysis in Part (b); it will actually be slower.

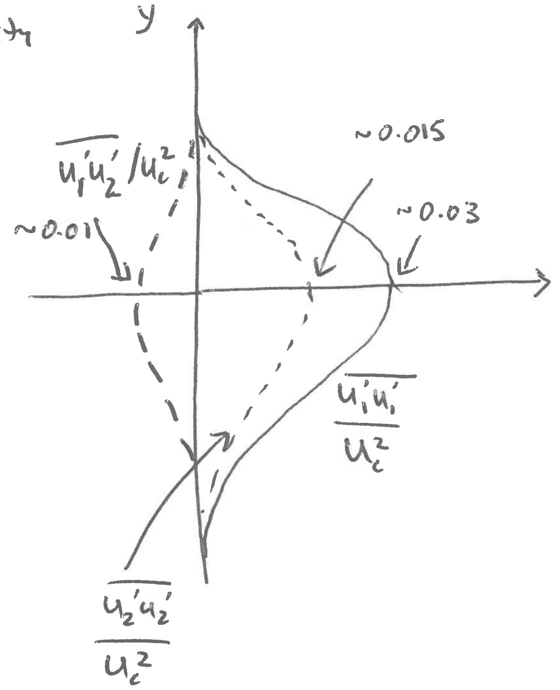
In our city traffic problem, we have a body-force generation mechanism, but this is limited to the surface, over a few m only associated with the vehicle's wake. We therefore have significant generation of turbulence only in part of our domain; the turbulence will have to diffuse upwards to fill the city's air up to height H. In reality, k will be much higher closer to where the traffic is and lower at larger heights above the ground. If you think of a wake behind each car, the characteristic velocity of the wake is V, which means that the characteristic turbulence intensity must be of the order of a fraction of V, around 0.1-0.2V, typical of thin shear flows.

Q4

(a) Self-preservation is a term that relates to the empirical finding, and the associated mathematical model, that in thin shear flows the turbulence and mean flow are always in an equilibrium mode at each downstream location, such that the turbulent velocity fluctuations are always a constant fraction of the characteristic velocity scale at each downstream location, and the large-eddy lengthscale scales with distance downstream. Cross-stream distributions, when normalised by the centreline values, look the same across all downstream locations. Hence we can write that for all mean quantities (stream wise velocity, cross stream velocity, Reynolds stresses, scalars etc) we have expressions such as: $U(x,y) = U_c(x)F_1\left(\frac{y}{\delta}\right)$; $V(x,y) = U_c(x)F_2\left(\frac{y}{\delta}\right)$; $\overline{u'_1u'_1}(x,y) = U_c(x)G_{11}\left(\frac{y}{\delta}\right)$; etc. , where δ is the characteristic thickness of the flow (δ depends on *x*).

(b) Key points in the diagrams below: (i) Mean velocity like an error function; (ii) streamwise turbulence higher than cross-stream components because the streamwise is the one that is produced by the shear, the other components are finite due to the pressure redistribution terms in the normal Reynolds stress equations; (iii) Reynolds stress is opposite to the sign of dU/dy and hence shown as -ve here.





(c) The typical large eddy lengthscale L will be a fraction of 0.1m, but not too small a fraction, say 0.4 x 0.1m, i.e. 0.04m. The characterictic turbulent velocity can be estimated to be of order 0.1-0.2U_c, with U_c=(20-10) m/s, i.e. u~2m/s. The order of magnitude of the dffusivity can be estimated as 0.1uL, hence 0.008 m^2/s (using the lower estimate of U/U_c). The kinematic viscosity of air at atmospheric conditions is $1.5x10^{-5}$ m^2/s, therefore the turbulence diffusivity is ~500 larger than the molecular one.