Version PAD/3

EGT3
ENGINEERING TRIPOS PART IIB

Monday 5 May 2014 2.00 to 3.30

Module 4A12

TURBULENCE AND VORTEX DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachments: 4A12 Data Card: (i) Vortex Dynamics (1 page);
(ii) Turbulence (2 pages).
Engineering Data Book

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

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1 (a) Sketch the secondary flow pattern in a Bödewadt layer and explain the physical origin of the secondary flow.

Use dimensional analysis to show that the boundary layer thickness is of the order of $\sqrt{\nu/\Omega}$, where ν is the fluid viscosity and Ω the rotation rate of the fluid. [35%]

(b) Water is pumped through a long helical duct of square cross-section. The axis of the helix is vertical and the pitch of the duct is so small that each turn may be considered as a flat, horizontal annulus. The duct has side W and the inner and outer radii of the duct are R and $R+W$. The flow is laminar and the primary motion is $u_\theta = \Omega r$ in cylindrical polar coordinates, where r is the distance from the axis of the helix.

Sketch the secondary flow pattern in the duct and use continuity to derive an estimate of the magnitude of the secondary flow in the core of the duct. [35%]

(c) Use an energy argument to explain why the pressure drop in the helix is dominated by Ekman pumping. What is the relationship between the pressure drop and the rate of destruction of energy in the Bödewadt layers? [30%]

2 (a) State Helmholtz's two laws of vortex dynamics. [20%]

(b) Consider a thin, isolated vortex tube in an inviscid fluid and a closed curve C that encircles the tube at $t = 0$. Use Kelvin's theorem to show that, if C is a material curve always composed of the same fluid particles, then C must encircle the vortex tube for all time. Hence show that vortex lines must move with the fluid, as if frozen into the fluid.

[30%]

(c) An alternative proof that the vortex lines are frozen into an inviscid fluid rests on the fact that a short line element, $d\mathbf{r}$, which links two material points in the fluid, is governed by the evolution equation

$$\frac{D}{Dt} d\mathbf{r} = (d\mathbf{r} \cdot \nabla)\mathbf{u} \quad .$$

Suppose that $d\mathbf{r}$ links two adjacent material points, A and B, that lie on the same vortex line at $t = 0$. Since $d\mathbf{r}$ and $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ are parallel at $t = 0$, we may write

$$d\mathbf{r}(t = 0) = \alpha\, \boldsymbol{\omega}(\mathbf{x}_{\mathrm{A}}, t = 0) \ ,$$

where α is a constant and $\mathbf{x}_A$ is the position vector of particle A.

Show that the vector

$$\mathbf{B}(\mathbf{x}_{\mathrm{A}}, t) = d\mathbf{r}(t) - \alpha\, \boldsymbol{\omega}(\mathbf{x}_{\mathrm{A}}, t)$$

satisfies

$$\frac{D\mathbf{B}}{Dt} = (\mathbf{B} \cdot \nabla)\mathbf{u}$$

and hence deduce that $\mathbf{B} = 0$ for $t \geq 0$.

Use the fact that $\mathbf{B} = 0$ to show that A and B must always lie on the same vortex line. [50%]

3 (a) Discuss briefly the *energy cascade* and the *inertial range* of the turbulent kinetic energy spectrum. [30%]

(b) If $E(\lambda)$ is the spectrum of the velocity fluctuations, where λ is the wavenumber, show that $E(\lambda) = C\varepsilon^{2/3}\lambda^{-5/3}$, where ε is the kinetic energy dissipation and C a constant. State clearly all assumptions made. [40%]

(c) Consider a turbulent flow with temperature fluctuations with spectrum $E_\theta(\lambda)$. By assuming that $E_\theta(\lambda)$ depends on ε, λ, and on the temperature fluctuation dissipation rate N, derive the spectral decay law for $E_\theta(\lambda)$. [30%]

4 (a) Discuss briefly the physical meaning of the various terms in the turbulent kinetic energy equation and state the order of magnitude of each one of these. [50%]

(b) Consider a homogeneous isotropic decaying turbulence and assume that the decay rates of the turbulent kinetic energy k and its dissipation ε are given by $k = C_1 t^{-p}$ and $\varepsilon = C_2 t^{-q}$, where C_1 and C_2 are constants and t is the time. By considering the $k-\varepsilon$ model as given in the Data Card, and given that $p = 5/4$, find how the integral length scale and eddy turnover time change with time. [50%]

END OF PAPER