Version PAD/3

EGT3
ENGINEERING TRIPOS PART IIB

Thursday 29 April 2021 9.00 to 10.40

Module 4A12

TURBULENCE AND VORTEX DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet and at the top of each answer sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachments: 4A12 Data Card: (i) Vortex Dynamics (1 page);
(ii) Turbulence (2 pages).
You are allowed access to the electronic version of the Engineering Data Books.

10 minutes reading time is allowed for this paper at the start of the exam.

The time taken for scanning/uploading answers is 15 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

1 (a) An axisymmetric vortex has a vorticity distribution in polar coordinates (r, θ, z) given by

$$\omega_z = \frac{\Phi_0}{\pi\delta^2} \exp\left(-r^2/\delta^2\right)$$

where the characteristic radius, δ, is a function of time and Φ_0 is a constant. Sketch the vorticity field and the associated velocity field. Show that

$$|\mathbf{u}| = \frac{\Phi_0}{2\pi r}\left[1 - \exp\left(-r^2/\delta^2\right)\right]$$

and state the physical significance of Φ_0. [15%]

(b) Show that the vortex in part (a) is an exact solution of the two-dimensional vorticity equation

$$\frac{\mathrm{D}\omega_z}{\mathrm{D}t} = \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \omega_z}{\partial r}\right)$$

provided that $d\delta^2/dt = 4\nu$. Find $\delta(t)$ and explain why δ grows in the way it does while Φ_0 is necessarily constant. [40%]

(c) This vortex is now subject to the axisymmetric strain field $u_r = -\frac{1}{2}\alpha r$ and $u_z = \alpha z$ where α is a constant strain rate.

(i) Show that this represents an exact solution of the axisymmetric vorticity equation

$$\frac{\mathrm{D}\omega_z}{\mathrm{D}t} = \omega_z \frac{\partial u_z}{\partial z} + \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \omega_z}{\partial r}\right),$$

provided that

$$\frac{d\delta^2}{dt} + \alpha\delta^2 = 4\nu \, .$$

(ii) For $\alpha = 4\nu/\delta^2$, which is Burgers' vortex, the flow is steady and δ is constant. Discuss the competing physical processes which balance in Burgers' vortex and lead to a steady flow.

(iii) What feature of Burgers' vortex makes it important for turbulence? [45%]

2 (a) State Kelvin's theorem and Helmholtz's two laws of vortex dynamics, noting that Helmholtz's second law comes in two parts. How would you expect Helmholtz's second law to be modified by viscosity for the case of an isolated vortex tube?

[20%]

(b) A short line element, $\mathrm{d}\mathbf{r}$, which links two material points in a fluid, is governed by

$$\frac{\mathrm{D}}{\mathrm{D}t}\mathrm{d}\mathbf{r} = \left(\mathrm{d}\mathbf{r}\cdot\nabla\right)\mathbf{u}$$

where $\mathbf{u}$ is the velocity field. Show that the material rate of change of the circulation $\Gamma = \oint_C \mathbf{u}\cdot\mathrm{d}\mathbf{r}$ in an inviscid fluid is

$$\frac{\mathrm{D}}{\mathrm{D}t}\oint_C \mathbf{u}\cdot\mathrm{d}\mathbf{r} = -\oint_C \nabla\left(p/\rho\right)\cdot\mathrm{d}\mathbf{r} + \oint_C \mathbf{u}\cdot\left((\mathrm{d}\mathbf{r}\cdot\nabla)\mathbf{u}\right)$$

where C is a *closed material curve*, p is the pressure and ρ the density. Hence prove Kelvin's theorem.

[20%]

(c) Use the evolution equation for $\mathrm{d}\mathbf{r}$ in part (b) to prove Helmholtz's first law. [20%]

(d) Consider an isolated vortex tube in an inviscid fluid which is encircled by a closed material curve C at $t = 0$. Explain why the flux of vorticity, Φ, is constant along the vortex tube and use Kelvin's theorem, along with Helmholtz's first law, to show that the flux Φ does not change with time as the vortex tube evolves. [20%]

(e) Use Helmholtz's second law to show that, if a short portion of a thin vortex tube is stretched so as to double its length, the mean vorticity in that part of the tube is also doubled. Hence explain how turbulence manages to maintain very high levels of vorticity.

[20%]

3 Consider turbulence in the air induced by traffic in a city. The city has length L, width W, and the height where atmospheric motions cease is H (*i.e.* we assume there is no exchange of matter or momentum for heights greater than H). There are a total of N cars, each assumed to move at constant speed V and have a drag coefficient C_D. For the purposes of this question, assume the direction of traffic to be random and equally distributed across all angles. Ignore any effects of buildings and consider the turbulence induced by the cars to be isotropic and homogeneous.

(a) If the turbulent length scale is $H/5$, determine the characteristic magnitude of the turbulent velocity fluctuations. State any assumptions made.

[30%]

(b) Suddenly, all traffic comes to rest. Derive an approximate expression for the subsequent time evolution of the turbulent kinetic energy k. State any assumptions made.

[40%]

(c) Assume you had access to a perfect instrument that measured the turbulent velocity fluctuations as a function of time. In reality, would the measured decrease of k in time be equal to, quicker or slower than your estimate in part (b)? Would k be homogeneous in the vertical direction? Explain your answers.

[30%]

4 (a) What do we mean by the term self-preservation for a thin shear flow? Discuss the underlying assumptions and write down general expressions for the mean and fluctuating velocities, the Reynolds stresses, and the scalar distributions. Give an example of a turbulent flow that is not self-preserving, stating your reasons.

[20%]

(b) Consider the shear layer formed between two streams downstream of a splitter plate where the free-stream velocities far from the layer are U_1 and U_2 respectively. What is the typical velocity scale in this problem? Sketch carefully the cross-stream distribution of the stream-wise and cross-stream mean velocities and of the Reynolds stresses, giving reasons for your choices.

[50%]

(c) At a particular distance downstream of the splitter plate, the characteristic thickness of the layer is 0.1 m. The free-stream velocities are U_1 = 20 ms^{-1} and U_2 = 10 ms^{-1}. Estimate the turbulent diffusivity and compare this with a typical molecular diffusivity. The fluid is air at atmospheric conditions.

[30%]

END OF PAPER

THIS PAGE IS BLANK

Vortex Dynamics Data Card

Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint \boldsymbol{A} \cdot d\boldsymbol{S}$

Stokes : $\int (\nabla \times A) \cdot d\boldsymbol{S} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$

Vector Identities

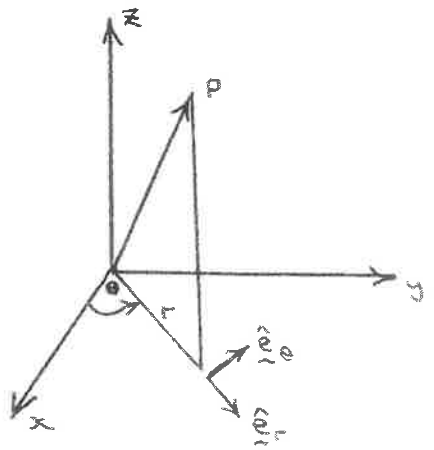
$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B}.\nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})$
$\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot \nabla f$
$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$
$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$

Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$



$$\nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)$$

Cambridge University Engineering Department

4A12: Turbulence

Data Card

Assume incompressible fluid with constant properties.

Continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

Mean momentum:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

Mean scalar:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D \frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

Turbulent kinetic energy ($k = \overline{u_i' u_i'}/2$):

$$\begin{aligned}
\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} \\
&\quad - \overline{u_i' u_j'} \frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'}
\end{aligned}$$

The $k - \varepsilon$ model:

$$\begin{aligned}
\frac{\partial k}{\partial t} + \overline{u}_i \frac{\partial k}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + P_k - \varepsilon \\
\frac{\partial \varepsilon}{\partial t} + \overline{u}_i \frac{\partial \varepsilon}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + c_{\varepsilon 1}\frac{\varepsilon}{k} P_k - c_{\varepsilon 2}\frac{\varepsilon^2}{k}
\end{aligned}$$

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

$$P_k = \frac{1}{2}\nu_t \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)^2$$

$$C_\mu = 0.09,\ c_{\varepsilon 1} = 1.44,\ c_{\varepsilon 2} = 1.92,\ \sigma_k = 1.0,\ \sigma_\varepsilon = 1.3$$

Energy dissipation:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

Scalar fluctuations ($\sigma^2 = \overline{\phi'\phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D\frac{\partial^2 \sigma^2}{\partial x_j^2} - 2\overline{\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'}\frac{\partial \overline{\phi}}{\partial x_j} - 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

Scalar fluctuations (modelled):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_i \frac{\partial \sigma^2}{\partial x_i} = \frac{\partial}{\partial x_i}\left((D + D_{turb})\frac{\partial \sigma^2}{\partial x_i}\right) + 2D_{turb}\left(\frac{\partial \overline{\phi}}{\partial x_i}\right)^2 - 2\overline{N}$$

Scalar dissipation:

$$2\overline{N} = 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2 = 2\frac{u}{L_{turb}}\sigma^2$$

Scaling rule for shear flow, flow dominant in direction x_1:

$$\frac{u}{L_{turb}} \sim \frac{\partial \overline{u}_1}{\partial x_2}$$

Kolmogorov scales:

$$\begin{aligned} \eta_K &= \left(\nu^3/\varepsilon\right)^{1/4} \\ \tau_K &= \left(\nu/\varepsilon\right)^{1/2} \\ v_K &= \left(\nu\varepsilon\right)^{1/4} \end{aligned}$$

Taylor microscale:

$$\varepsilon = 15\nu\frac{u^2}{\lambda^2}$$

Eddy viscosity (general):

$$\begin{aligned} \overline{u_i' u_j'} &= -\nu_{turb}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij} \\ \overline{u_j' \phi'} &= -D_{turb}\frac{\partial \overline{\phi}}{\partial x_j} \end{aligned}$$

Eddy viscosity (for simple shear):

$$\overline{u_1' u_2'} = -\nu_{turb}\frac{\partial \overline{u}_1}{\partial x_2}$$