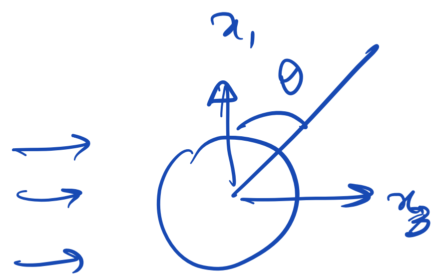
1. The conservation eq^{n}s can be written as



$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \underline{u}' = 0 \quad (1)$$

$$\rho_0 \frac{\partial \underline{u}'}{\partial t} + \nabla p' = \psi \, \underline{e}_{x_1} \quad (2)$$

force per unit volume

$\frac{\partial}{\partial t}(1) - \nabla\cdot(2)$ gives:

$$\frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 \underbrace{p'}_{c_0^2 \rho'} = - \frac{\partial \psi}{\partial x_1}(\underline{x}, t)$$

Using the Green's f^n:

$$\rho' = - \iint \frac{\partial \psi}{\partial y_1}(\underline{y}, \tau) \frac{\delta(\underline{x} - \underline{y} - c_0(t-\tau))}{4\pi c_0 |\underline{x} - \underline{y}|} \, d\underline{y} \, d\tau$$

$$= - \frac{\partial}{\partial x_1} \iint \psi(\underline{y}, \tau) \frac{\delta(\underline{x} - \underline{y} - c_0(t-\tau))}{4\pi c_0 |\underline{x} - \underline{y}|} \, d\underline{y} \, d\tau$$

Carrying out the τ integral,

$$\rho' = -\frac{\partial}{\partial x_1}\int \frac{\psi\left(\underline{y},\, t-\frac{|\underline{x}-\underline{y}|}{c_0}\right)}{4\pi c_0^2|\underline{x}-\underline{y}|}\,d\underline{y}$$

Because the source is compact, assume $\underline{y} \sim 0$ above in the argument of ψ:

$$\rho' = -\frac{\partial}{\partial x_1}\int \frac{\psi\left(\underline{y},\, t-\frac{r}{c_0}\right)}{4\pi c_0^2|\underline{x}-\cancel{\underline{y}}|}\,d\underline{y}$$

0 (far-field).

$$\therefore \rho' \sim -\frac{1}{4\pi c_0^2 r}\frac{\partial}{\partial x_1}\left(fL\cos\left(\omega\left(t-\frac{r}{c_0}\right)\right)\right)$$

$$= -\frac{fL}{4\pi c_0^2 r}(-)\omega\sin\left(\omega\left(t-\frac{r}{c_0}\right)\right)\cdot\left(-\frac{1}{c_0}\right)\cos\theta$$

$$= -\frac{fL\omega}{4\pi c_0^3 r}\sin\omega\left(t-\frac{r}{c_0}\right)\cos\theta$$

$p' = c_0^2 \rho'$

$$\therefore \boxed{p' = -\frac{1}{4\pi}\frac{L\cos\theta}{r}\frac{\omega}{c_0} f \sin\omega\left(t - \frac{r}{c_0}\right)}$$

(b) In the far field, $\frac{p'}{u'} \sim \rho_0 c_0$

$$\therefore I = p'u' = \frac{p'^2}{\rho_0 c_0}$$

$$\bar{I} = \frac{1}{2}\frac{\hat{p}\hat{p}^*}{\rho_0 c_0}$$

$$p' = \mathrm{Im}\left\{\underbrace{-\frac{1}{4\pi}\frac{L\cos\theta}{r}\frac{\omega}{c_0} f e^{-i\omega\frac{r}{c_0}}}_{\hat{p}} e^{i\omega t}\right\}$$

$$\therefore \bar{I} = \frac{1}{2\rho_0 c_0}\frac{1}{(4\pi)^2}\frac{L^2\cos^2\theta}{r^2}\frac{\omega^2}{c_0^2} f^2$$

$$\bar{P} = \int \bar{I}\, ds$$

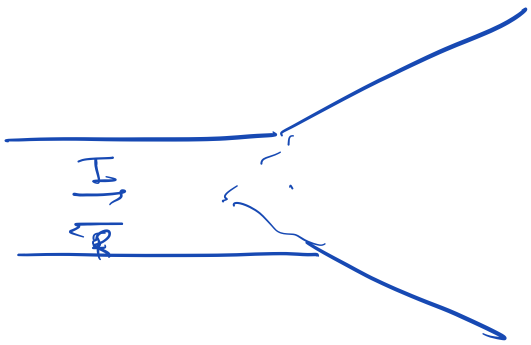
$$= \int_0^{\pi} \bar{I}\, 2\pi r^2 \sin\theta \, d\theta$$

$$= \frac{\pi}{\rho_0 c_0 (4\pi)^2} \frac{L^2 \cancel{r^2}}{\cancel{r^2}} \frac{\omega^2 f^2}{c_0^2} \left(\frac{2}{3}\right)$$

$$= \frac{1}{24\pi} \frac{\omega^2 L^2 f^2}{\rho_0 c_0^3}$$

(c) L must be small enough so that the source ~~can~~ can be treated as compact.

2



(a) $$p' = \frac{f(t - r/c_0)}{r}$$

$$= \frac{e^{i\omega(t - r/c_0)}}{r}$$

$$\rho_0 i\omega u_r' = +\left\{\frac{1}{r^2} + \frac{i\omega}{c_0 r}\right\} e^{[\ ]}$$

$$u_r' = \frac{1}{\rho_0 i\omega r^2}\left\{1 + ikr\right\} e^{\{\ \}}$$

$$Z = \frac{p'}{u'} = \frac{i r_0 \omega \rho_0}{(1 + ikr_0)} = \underline{\underline{\frac{\rho_0 c_0 ikr_0}{1 + ikr_0}}}$$

(b)

$$p' = \left\{ I e^{-ikx} + R e^{ikx} \right\} e^{i\omega t}$$

$$u' = \frac{1}{\rho_0 c_0} \left\{ I e^{-ikx} - R e^{ikx} \right\} e^{i\omega t}$$

$x = 0$.

$$p' = (I + R) e^{i\omega t}$$

$$u' = \frac{(I - R)}{\rho_0 c_0} e^{i\omega t}$$

$$\frac{p'}{u'} = \frac{(I + R) \rho_0 c_0}{I - R}$$

$$\frac{(I+R)\,\cancel{\rho_0 c_0}}{(I-R)} = \frac{ikr_0\,\cancel{\rho_0 c_0}}{(1+ikr_0)}$$

$$\frac{\frac{I}{R}+1}{\frac{I}{R}-1} - 1 = \frac{ikr_0}{1+ikr_0} - 1$$

$$\Rightarrow \quad \frac{2}{\frac{I}{R}-1} = \frac{-1}{1+ikr_0}$$

$$\frac{\frac{I}{R}-1}{2} = -\left(\frac{1+ikr_0}{2}\right)$$

$$\frac{I}{R} = -2(1+ikr_0)+1$$

$$= -2ikr_0 - 1$$

$$\frac{R}{I} = -\frac{1}{1+2ikr_0}$$

(c) In the absence of the horn,

$$\frac{R}{I} = -1$$

Therefore the horn reduces the amplitude of the reflected waves and hence aids transmission.

The efficiency of the sound transmission is frequency dependent.

If $kr_0 \ll 1$, the horn is not effective at all. However, if $kr_0 \gg 1$,

$$\frac{R}{I} \sim 0$$

i.e. we get perfect transmission.

Can you explain this result from a physical point of view?

Hint: think impedance and plane waves.

3 (a) An acoustic mode in a duct is said to be 'cut-off' if it is evanescent. Then the axial wavenumber is purely imaginary so that the amplitude of the disturbance decays exponentially along the duct.

(b) The formula on the data card gives the pressure field as

$$p'(\underline{x},t) = e^{i(\omega t + n\theta)} J_n\left(\frac{z_{mn} r}{a}\right)\left(A e^{-ikx_3} + B e^{ikx_3}\right)$$

where z_{mn} is the mth zero of $\dot{J}_n(z)$ and $k = (k_o^2 - z_{mn}^2/a^2)$, $k_o = \omega/c_o$.

We denote the rotation rate by Ω.

Blade passing frequency is 3Ω

At harmonics of blade passing frequency $\omega = 3M\Omega$ M integer
$n = 3M$,

for cut-off modes we need k imaginary ie. $k_o < \dfrac{z_{mn}}{a}$

$$\frac{3M\Omega}{c_o} < \frac{z_{1\,3M}}{a}$$

For $M = 1$, we require $\quad \Omega < \dfrac{c_o}{3a} z_{13} = \dfrac{c_o}{3a}\,4.20119$

$= 2 \quad \Omega < \dfrac{c_o}{3a}\dfrac{z_{16}}{2} = \dfrac{c_o}{3a}\dfrac{7.50129}{2}$

$= \geq 3 \quad \Omega < \dfrac{c_o}{3a}\dfrac{z_{1\,3M}}{M} = \dfrac{c_o}{3a}\left(\dfrac{3M + 0.80861(3M)^{1/3}}{M}\right)$

$$< \frac{c_o}{a}\left(1 + \frac{0.80861}{3^{2/3}} M^{-2/3}\right)$$

It is clear that the most stringent constraint is for large M and that we need $\quad \Omega < \dfrac{c_o}{a} = \dfrac{343}{0.15} = 2.2867 \times 10^3 \text{ rads}^{-1}$

$$= \underline{\underline{21{,}836 \text{ rpm}}}$$

(c) The rotor pattern has the form $e^{iM3(\Omega t - \theta)}$

and the stator pattern is $e^{\mp iNS\theta}$ where S = number of stator blades

So rotor-stator interaction modes are: $e^{iM3(\Omega t - \theta) \mp iNS\theta}$

$$= e^{i(M3\Omega t - (M3 \pm NS)\theta)}$$

Modes propagate if $\quad \dfrac{3M\Omega a}{c_o} > z_{1\;3M\pm NS}$

Usually $S \simeq 2.4 \times$ number of rotor blades and S prime is a good choice.

Here $2.4 \times 3 = 7.2$ so try $S = 7$, $S = 11$ is another possibility.

For 10,000 rpm $\frac{\Omega a}{c_0} = 0.458 < 1$ so all rotor-alone modes are cut-off

Mode of frequency $M \times$ bpf propagate if $0.458 \times 3M > z_{1|3M \mp N7|}$

i.e. if $0.458 > \frac{z_{1|3M \mp N7|}}{3M}$

For 7 stator blades

For bpf, $M = 1$

$$\frac{z_{1|3 \mp N7|}}{3} > 1 \quad \text{for all } N \text{ and so certainly } > 0.458$$

<u>All modes at bpf are cut-off</u>

For 2bpf, $M = 2$

$$\frac{z_{1|6 \mp N7|}}{6} \text{ is } > 1 \quad \text{for all } N \text{ except } N = 1$$

For $N=1$

$|6 - 7| = 1$

$$\frac{z_{11}}{6} = \frac{1.84118}{6} = 0.3068 < 0.458$$

So <u>the first order stator mode propagates at 2 bpf with $S=7$</u>

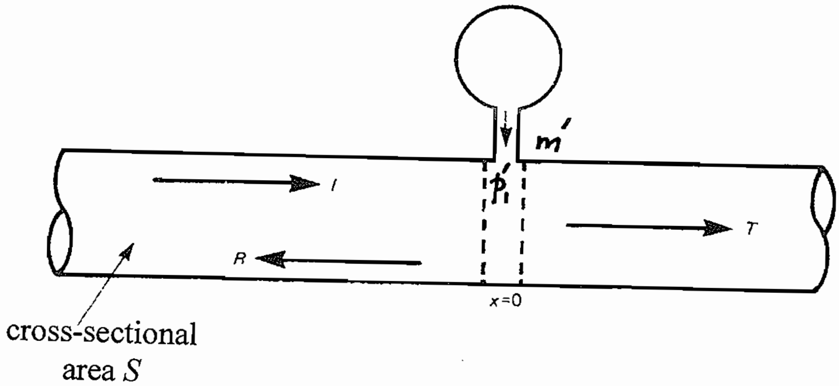
A choice of $S = 11$ would have been better for 1 and 2 bpf,

Then the minimum value of $|6 \mp N11|$ is 5 and $z_{15} = 6.41562$

$$\frac{z_{15}}{6} = \frac{6.41562}{6} > 0.458$$

and all modes are cut-off at <u>2 bpf as well as bpf.</u>

4



Continuity of pressure at $x=0$

$$(I+R)e^{i\omega t} = p_1' = Te^{i\omega t} \qquad (1)$$

Continuity of mass flow rate

$$\frac{(I-R)}{c_0} S e^{i\omega t} + m' = \frac{TS}{c_0} e^{i\omega t} \qquad (2)$$

where m' = mass flow rate out of Helmholtz resonator.

From (1), $R = T - I$ and substituting into (2) gives

$$(2I-T)\frac{S}{c_0} e^{i\omega t} + m' = \frac{TS}{c_0} e^{i\omega t} \qquad (3)$$

From equation (1.20) in the Lecture Notes (a full solution would include deriving the Helmholtz resonator result as in Lecture Notes)

$$m' = \frac{i\omega A}{\ell} \frac{1}{\omega^2 - c_0^2 A/V\ell} p_1'$$

$$= \frac{i\omega A}{\ell} \frac{1}{\omega^2 - c_0^2 A/V\ell} Te^{i\omega t} \quad \text{from equation (1)}$$

Substituting for m' in equation (3) leads to

$$(2I-T)\frac{S}{c_0} + \frac{i\omega A}{\ell} \frac{1}{\omega^2 - c_0^2 A/V\ell} T = \frac{TS}{c_0}$$

i.e. $$2I = 2T + \frac{i\omega A c_0}{S\ell} \frac{1}{\omega^2 - c_0^2 A/V\ell} T$$

Ratio of incident to transmitted acoustic power $= \dfrac{|I|^2}{|T|^2}$

$$= \frac{1}{1 + \left(\dfrac{\omega A c_0}{2S\ell} \dfrac{1}{\omega^2 - c_0^2 A/V\ell}\right)^2} = \frac{1}{1 + \dfrac{1}{4S^2}\left(\dfrac{1}{\dfrac{\omega \ell}{c_0 A} - \dfrac{c_0}{\omega V}}\right)^2}$$